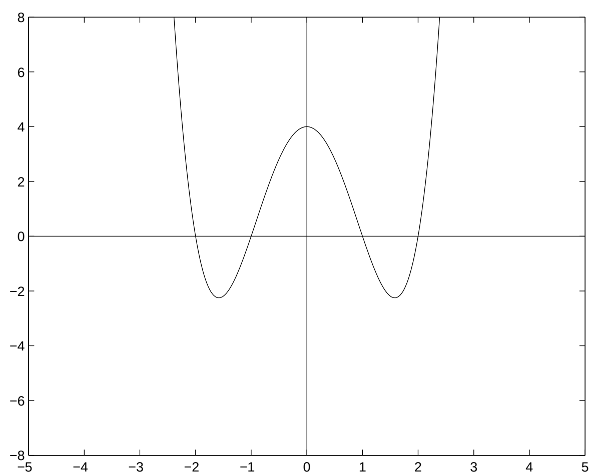
Homework 9 Foundations of Computational Math 1 Fall 2011

Problem 9.1

Let $f(x) : \mathbb{R} \rightarrow \mathbb{R}$ be given by

$$f(x) = x^4 - 5x^2 + 4$$

and consider applying Newton's method for optimization. Here Newton's method refers to the basic form where the step size is 1 and nothing is done to alter the Hessian to guarantee positive definiteness. Note that $f(x)$ is a scalar function of a scalar argument and has the form



(i) What are the values of x that are local minimizers or local maximizers of $f(x)$. Justify your answers.

(ii) Find the value $\beta > 0$ such that $f(x)$ has negative curvature for $-\beta < x < \beta$, and positive curvature outside the interval, i.e., for $x < -\beta$ or $x > \beta$.

(iii) What happens to the Newton step at $x = \beta$?

(iv) Determine $\mu(x) : \mathbb{R} \rightarrow \mathbb{R}$ such that the step of Newton's method applied to $f(x)$ can be written as $x_{k+1} = \mu(x_k)x_k$.

(v) Find the value of $\alpha \in \mathbb{R}$ such that $\beta > \alpha > 0$ and Newton's method cycles and does not converge when $x_0 = \alpha$ or $x_0 = -\alpha$. That is, $-\alpha = \mu(\alpha)\alpha$ and $\alpha = -\mu(-\alpha)\alpha$.

(vi) Show that if $-\alpha < x < \alpha$ then

$$|\mu(x)| < 1$$

(vii) Show that if $-\alpha < x_0 < \alpha$ is the initial point for Newton's method then there is a constant $0 < \gamma < 1$ (possibly dependent on x_0 but independent of k) such that

$$|x_{k+1}| < \gamma |x_k|$$

and therefore $x_k \to 0$.

(viii) It can be shown that if $x_0 > \sigma > 0$ where σ is the rightmost local minimizer of $f(x)$ then $x_k \to \sigma$ for Newton's method. Using this fact and those above, show that it is possible to choose $-\beta < x_0 < -\alpha$ so that $x_k \to \sigma$ for Newton's method.

(ix) Implement Newton's method (in MATLAB preferably) and demonstrate the convergence behavior determined above.