Solutions to Program 4 Foundations of Computational Math 1 Fall 2011

1 General Task

In this assignment you will implement solvers for nonlinear systems and apply them to a specific system of 2 nonlinear equations. You will also analyze some of your iterations to show that the must converge.

2 Submission of Results

Expected results comprise:

1. A document giving the analyses requested below and the describing the results of the experiments required below.
2. The source code, makefiles, and instructions on how to compile and execute your code including the math department's machine used if applicable.
3. Code documentation should be included in each routine.

These results should be emailed to gallivan@math.fsu.edu by 11:59PM on the due date above. You may be asked to demonstrate your code if your document does not completely convince me that you tested your code sufficiently.

3 Specific Tasks

3.1 The Equations

Consider the system of equations in $\mathbb{R}^2$

$$\xi^2 + \eta^2 = 4$$
$$e^{\xi} + \eta = 1$$

Show that the system has two solutions in $\mathbb{R}^2$, one with $\xi > 0$ and $\eta < 0$ and one with $\xi < 0$ and $\eta > 0$.

Solution:

It is easy to see that the two equations are a circle of radius 2 center at the origin and exponential opening down and to the left. The exponential, passes through the origin, has an asymptote at $\eta = 1$ as $\xi \to -\infty$, and is such that $\eta \to -\infty$ as $\xi \to \infty$. Therefore, there must be two intersection points: one with $\xi > 0$ and $\eta < 0$ and one with $\xi < 0$ and $\eta > 0$.

The leftmost one with $\xi_\ell < 0$ and $\eta_\ell > 0$ can be isolated further using the exponential's horizontal asymptote and the finite extent of the circle and the intersection of the asymptote with the circle. It must satisfy

$$-2 < \xi_\ell < -\sqrt{3} \quad \text{and} \quad 0 < \eta_\ell < 1$$

The rightmost one with $\xi_r > 0$ and $\eta_r < 0$ can be isolated further using the finite extent of the circle and the intersection of the exponential with the line $\eta = -2$, i.e., $\xi = \ln 3$. It must satisfy

$$0 < \xi_r < \ln 3 \approx 1.09 \quad \text{and} \quad -2 < \eta_r < 0$$

Also note that if we set $\eta = -\sqrt{2}$ we have

$$\ln(1 + \sqrt{2}) < \sqrt{2}$$

so the exponential is still inside the circle and we have

$$0 < \xi_r < \ln 3 \approx 1.09 \quad \text{and} \quad -2 < \eta_r < -\sqrt{2}$$

These bounds could be refined but that amounts to using a bisection-like method to solve the problem analytically. The regions could also be used to choose potential initial points for the iterations below but we will consider those choices more widely over $\mathbb{R}^2$.

3.2 Iteration 1

Consider the following function and iteration:

$$G_1(x) = \begin{pmatrix} \gamma_1(\xi, \eta) \\ \gamma_2(\xi, \eta) \end{pmatrix} = \begin{pmatrix} \ln(1 - \eta) \\ -\sqrt{4 - \xi^2} \end{pmatrix}$$

$$\begin{pmatrix} \xi_{k+1} \\ \eta_{k+1} \end{pmatrix} = \begin{pmatrix} \gamma_1(\xi_k, \eta_k) \\ \gamma_2(\xi_k, \eta_k) \end{pmatrix} = \begin{pmatrix} \ln(1 - \eta_k) \\ -\sqrt{4 - \xi_k^2} \end{pmatrix}$$

1. Note that all iterates must remain real. Consider where in the $\mathbb{R}^2$ a value of ξ_k or η_k would cause complex values to appear on the next iteration.

2. Find a domain for the initial condition $(\xi_0,\ \eta_0)$ such that the iteration converges to one of the roots. Prove your assertion.

3. Can this iteration converge to either root by appropriate choice of $(\xi_0,\ \eta_0)$? Prove your assertion.

4. Implement the iteration and verify with several initial conditions in the domain that iteration converges to the predicted root.

Solution: To keep the iteration real note that

$$-2 \le \xi_k \le 2 \to \eta_{k+1} \in \mathbb{R}, \quad \eta_{k+1} \le 0$$
$$\eta_k < 0 \to \xi_{k+1} \in \mathbb{R}, \quad \xi_{k+1} > 0$$
$$0 \le \eta_k < 1 \to \xi_{k+1} \in \mathbb{R}, \quad \xi_{k+1} \le 0$$

So $0 \le \eta_k < 1$ we have $\xi_{k+1} < 0$ and we must not let it go smaller than -2. Therefore keeping $\eta_k \le 0$ is reasonable. To further keep the value of η_k from pushing ξ_{k+1} out of its desired range we can take

$$-6.4 \approx 1 - e^2 < \eta_k < 0$$

Putting all of this together we have

$$-6.4 \approx 1 - e^2 < \eta_k < 0 \to 0 < \xi_{k+1} < 2 \to -2 < \eta_{k+2} < 0$$

$$\therefore 0 < \xi_0 < 2 \quad \text{and} \quad -2 < \eta_0 < 0$$

keeps the iteration real and remains in this domain, i.e.,

$$0 < \xi_k < 2 \quad \text{and} \quad -2 < \eta_k < 0$$

Therefore, there is no chance of the iteration reaching the root at (ξ_ℓ, η_ℓ) if starting in this domain.

We must now determine a subdomain in which the iteration converges to (ξ_r, η_r). The Jacobian of $G_1(x)$ is

$$J_1(x) = \begin{pmatrix} 0 & \frac{1}{1-\eta} \\ \frac{\xi}{\sqrt{4-\xi^2}} & 0 \end{pmatrix}$$

Clearly, $1/3 < 1/1 - \eta < 1$ for $-2 < \eta < 0$. We have

$$\frac{\xi}{\sqrt{4-\xi^2}} = \begin{cases} 0 & \text{when } \xi = 0 \\ 1 & \text{when } \xi = \sqrt{2} \\ \to \infty & \text{when } \xi \to 2 \end{cases}$$

It is an increasing function of ξ so if we further restrict our domain for $(\xi,\ \eta)$ to

$$0 < \xi < \sqrt{2} \quad \text{and} \quad -2 < \eta < 0$$

we have by Gershgorin $\rho(J_1(x)) < 1$ and a convergent iteration to the root $(\xi_r,\ \eta_r)$. Also note that given the bound $-2 < \xi_\ell < -\sqrt{3}$ and $0 < \eta_\ell < 1$ we know that $\rho(J_1(x_\ell)) > 1$ and the leftmost root is a point of repulsion so $G_1(x)$ cannot converge to x_ℓ for any initial condition. (Look at the determinant of $J_1 - \lambda I$.)

Note this is a sufficient condition. The iteration may converge to the root $(\xi_r,\ \eta_r)$ from other points as well.

Take care when implementing this iteration in MATLAB. MATLAB will use complex values for logarithms when the arguments are negative. The iteration will then proceed with complex iterates with no overt warning, i.e., you must monitor this if you do not control your initial conditions to guarantee a real iteration.

Also note that the fact that $G_1(x)$ (and as is shown below $G_2(x)$) converges to only one of the roots is not necessarily undesireable. If only one of the roots is of interest then designing an iteration that is repulsed by roots that are not of interest is valuable.

3.3 Iteration 2

Consider the following function and iteration:

$$G_2(x) = \begin{pmatrix} \gamma_1(\xi,\eta) \\ \gamma_2(\xi,\eta) \end{pmatrix} = \begin{pmatrix} -\sqrt{4-\eta^2} \\ 1-e^{\xi} \end{pmatrix}$$

$$\begin{pmatrix} \xi_{k+1} \\ \eta_{k+1} \end{pmatrix} = \begin{pmatrix} \gamma_1(\xi_k,\eta_k) \\ \gamma_2(\xi_k,\eta_k) \end{pmatrix} = \begin{pmatrix} -\sqrt{4-\eta_k^2} \\ 1-e^{\xi_k} \end{pmatrix}$$

1. Note that all iterates must remain real. Consider where in the $\mathbb{R}^2$ a value of ξ_k or η_k would cause complex values to appear on the next iteration.
2. Find a domain for the initial condition $(\xi_0,\ \eta_0)$ such that the iteration converges to one of the roots. Prove your assertion.
3. Can this iteration converge to either root by appropriate choice of $(\xi_0,\ \eta_0)$? Prove your assertion.
4. Implement the iteration and verify with several initial conditions in the domain that iteration converges to the predicted root.

We have

$$-2 < \eta_k < 2 \rightarrow -2 < \xi_k < 0 \rightarrow 0.865 \approx 1 - e^{-2} < \eta_{k+1} < 1$$

Therefore we can take

$$-2 < \xi < 0 \quad \text{and} \quad -2 < \eta < 2$$

to define a region of real iteration. Therefore, there is no chance of the iteration reaching the root at (ξ_r, η_r) if starting in this domain.

The Jacobian of $G_2(x)$ is

$$J_2(x) = \begin{pmatrix} 0 & \frac{\eta}{\sqrt{4-\eta^2}} \\ -e^{\xi} & 0 \end{pmatrix}$$

We have

$$-2 < \xi < 0 \to -1 < -e^{\xi} < -e^{-2} \approx -0.1353$$

and

$$0 < \eta < \sqrt{2} \to 0 < \frac{\eta}{\sqrt{4-\eta^2}} < 1$$

and we have by Gershgorin $\rho(J_2) < 1$ and a convergent iteration to the root $(\xi_\ell,\ \eta_\ell)$.

Also note that using the bounds on the root $(\xi_r,\ \eta_r)$ we know that $\rho(J_2(x_r)) > 1$ and the leftmost root is a point of repulsion so $G_2(x)$ cannot converge to x_r for any initial condition.