Set 5: Polynomial Interpolation – Part 4

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Convergence of Interpolating Polynomials

- convergence of polynomials
- interpolation strategies
- convergence of interpolation strategies
Convergence on Interval

Approximation by polynomials is motivated by the following theorem:

**Theorem 5.1.** *(Weierstrass Approximation Theorem)* If \( f(x) \in C^{(0)}[a, b] \) then \( \forall \epsilon > 0 \ \exists n \in \mathbb{Z} \) and polynomial \( p_n(x) \) with degree at most \( n \) such that

\[
\| f(x) - p_n(x) \|_\infty < \epsilon.
\]

This is uniform convergence, i.e., pointwise error at all points in interval is bounded and the bound is going to 0.
Convergence on Interval

- Theorem 5.1 gives no insight into how to choose $p_n(x)$ and does not relate necessarily to an interpolation strategy.
- The result can be derived as a corollary to a constructive theorem due to Bernstein.
- A sequence of polynomials is defined and shown to converge uniformly.
Bernstein Polynomials

**Definition 5.1.** Let $f(x)$ be a real function defined on $[0, 1]$. The $n$-th Bernstein polynomial for $f$ is

$$B_n(x) = B_n(x; f) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k}$$

$$= \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \phi_{n,k}(x)$$

$$= \sum_{k=0}^{n} f(x_k) \phi_{n,k}(x)$$

where $x_k = k/n$.
Bernstein Polynomials

- Sum of $f(x)$ at uniformly-spaced points.
- The weight $\phi_{n,k}(x)$ is non-negative on $[0, 1]$ and $\sum_{k=0}^{n} \phi_k(x) = 1$.
- The weight $\phi_{n,k}(x)$ can be very small for $k$ where $x$ is far from $k/n$.
- The weight $\phi_{n,k}(x)$ achieves its maximum on $[0, 1]$ at $x = k/n$.
- The construction is not interpolatory, i.e., $B_n(x_k)$ is not necessarily equal to $f(x_k)$.
- $B_n(x)$ usually interpolates $f(x)$ but where and how often it does is not controlled.
**Bernstein Approximation**

**Theorem 5.2.** If \( f(x) \in C^0[0, 1] \) then \( B_n(x) \) converges uniformly to \( f(x) \) on \([0, 1] \), i.e.,

\[
\lim_{n \to \infty} \|f(x) - B_n(x)\|_\infty = 0
\]

**Proof.** See Bartle, Elements of Real Analysis (1976)

**Corollary 5.3.** If, in addition, on \([0, 1] \), \( f(x) \) satisfies the Lipschitz condition \(|f(x) - f(\hat{x})| < \lambda|x - \hat{x}|\) then

\[
\|f(x) - B_n(x)\|_\infty < \frac{9}{4}\lambda n^{-1/2}
\]

**Proof.** See Isaacson and Keller (1966)
Bernstein Approximation

- Easily updated to apply to $[a, b]$.
- Convergence is much slower than other approximation methods.
- Even if $f(x) \in C^{(p)}[0, 1]$ with $p \geq 2$ convergence remains relatively slow.
- Useful theoretical result but Bernstein polynomials are not used in practice for this type of approximation.
- Bernstein polynomials are used when “shape” is important.
- This shows that polynomials **can converge uniformly** to a continuous $f$. 
$f(x) = 1/(1 + 10x^2) - 1 \leq x \leq 1$ shifted to $[0, 1]$ – black, $B_3(x)$ – red, $B_6(x)$ – blue, $B_{15}(x)$ – green
Convergence of Interpolating Polynomials

Definition 5.2. An interpolating strategy is defined by a sequence, $X$, of sets of nodes $X_n = \{x^{(n)}_0, \ldots, x^{(n)}_n\}$.

- The sets $X_n$ are chosen independently of any particular $f(x)$.
- Each $X_n$ defines an interpolatory polynomial, $p_n(x)$, of degree $n$ such that given an $f(x)$, $p_n(x^{(n)}_i) = f(x^{(n)}_i)$ for $0 \leq i \leq n$.

Uniform interpolation:

$$X_n = \{x^{(n)}_i = x_0 + ih, \quad h = (b - a)/n\}$$

Chebyshev interpolation:

$$X_n = \{x^{(n)}_j = \cos\left(\frac{2j + 1}{n + 1} \pi\right)\}$$
Convergence of Interpolating Polynomials

The convergence of

$$\|f(x) - p_n(x)\|_{\infty}$$

on a closed interval $[a, b]$ for $f(x) \in C^{(0)}[a, b]$ is complicated.

The result depends on

- the choice of $X$,
- the class of functions $f(x)$ that may be more constrained than $C^{(0)}[a, b]$. 
Runge’s Phenomenon

Let $I = [-5, 5]$ and define $x_j^{(n)} = -5 + jh_n$ with $h_n = 10/n$ and $0 \leq j \leq n$. The sets $X_n = \{x_0^{(n)}, \ldots, x_n^{(n)}\}$ define a sequence, $X$, of sets of nodes each of which define an interpolatory polynomial, $p_n(x)$, of degree $n$. It can be shown that

$$\lim_{n \to \infty} \|f(x) - p_n(x)\|_\infty$$

does not converge on $I$ for $f(x) = 1/(1 + x^2)$.

Proof. See Isaacson and Keller (1966)
Runge’s Phenomenon

\[ f(x) = \frac{1}{1 + x^2} \] – black, \( p_5(x) \) – blue, \( p_{10}(x) \) – red
Runge’s Phenomenon

- The divergence occurs near the endpoints of the interval.
- This is typical behavior so keep order low to be effective with uniformly spaced points.
- Non-uniform points more dense near endpoints are needed for better interpolation strategies, e.g., Chebyshev.
Convergence of Interpolating Polynomials

For each degree $n$ we can define the “best” polynomial approximation:

**Definition 5.3.** Let $p_n^*(x) \in \mathbb{P}_n$ be such that
$$E_n^* = \| f(x) - p_n^*(x) \|_{\infty} \leq \| f(x) - q_n(x) \|_{\infty} \quad \forall q_n(x) \in \mathbb{P}_n.$$  

This approximation will be discussed in much more detail later.
Convergence of Interpolating Polynomials

Lemma. Let the sequence $X$ define an interpolating strategy, and let the Lebesgue constant be

$$
\Lambda_n(X) = \| \sum_{j=0}^{n} |\ell_j(n)(x)| \|_{\infty}
$$

for the set of nodes $X_n = \{x_0^{(n)}, \ldots, x_n^{(n)}\}$ where $\ell_j(n)(x)$ are the Lagrange characteristic functions associated with $X_n$.

If $f(x) \in C^{(0)}[a, b]$ then

$$
E_n^* \leq \| f(x) - p_n(x) \|_{\infty} \leq (1 + \Lambda_n(X))E_n^*
$$

for $n = 0, 1, \ldots$. 

Convergence of Interpolating Polynomials

- A small Lebesgue constant $\Lambda_n(X)$ guarantees good $\infty$ norm approximation of $f(x)$ for the associated $p_n(x)$.
- Bounding the Lebesgue constant $\Lambda_n(X)$ is a key task when analyzing an interpolating strategy.
- Erdos (1961) showed $\forall X \exists C > 0$ such that
  \[
  \Lambda_n(X) > \frac{2}{\pi} \log(n + 1) - C \quad n = 0, 1, \ldots
  \]
  so $\Lambda_n(X) \to \infty$.
- Natanson (1965) showed for equally spaced nodes
  \[
  \Lambda_n(X) \approx \frac{2^{n+1}}{en \log n}
  \]
Convergence of Interpolating Polynomials

- The error bound predicted by the Lebesgue constant is not achieved for all $f(x) \in C^{(0)}[a, b]$.
- A particular strategy may work well with a particular $f$ or some particular class of $f$.
- Unfortunately, no interpolating strategy, $X$, converges for all $f(x) \in C^{(0)}[a, b]$. 
Convergence of Interpolating Polynomials

**Theorem 5.4.** (Faber 1914) Given an interpolating strategy defined by any sequence of node sets $X$ on $[a, b]$, $\exists f(x) \in C^{(0)}[a, b]$ such that $\|f(x) - p_n(x)\|_\infty$ does not converge.
Summary

- (Bernstein) $B_n(x)$ converge uniformly for all $f(x) \in C^0[0,1]$ but not an interpolating strategy since the number and position of points where they agree with $f(x)$ depend on $f(x)$.

- (Faber) No $p_n(x)$ defined by an $X$ converges for all $f(x) \in C^0[0,1]$.

- (Bernstein) and (Brutman, Passow) interpolant for $|x|$ on $[-1,1]$ diverges almost everywhere for a variety of well-known node sets.

- For an interpolating strategy to converge uniformly:
  - the class of $f(x)$ is more restrictive than $C^0[0,1]$,
  - the nodes in $X_n = \{x_0^{(n)}, \ldots, x_n^{(n)}\}$ are chosen carefully.
Uniform Convergence of Interpolating Polynomials

**Theorem 5.5.** Let $I = [-1, 1]$ and let the interpolating strategy be defined by the sets $X_n = \{x_0^{(n)}, \ldots, x_n^{(n)}\}$ given by the Chebyshev zeros

$$x_j^{(n)} = \cos\left(\frac{2j + 1}{n + 1} \frac{\pi}{2}\right) \quad 0 \leq j \leq n.$$

- If $f(x) \in C^{(2)}[I]$ then $\|f(x) - p_n(x)\|_\infty$ converges uniformly on $I$.

- If $f(x) \in C^{(0)}[I]$ satisfies the Lipschitz condition

$$|f(x) - f(\hat{x})| < \lambda |x - \hat{x}|$$

then $\|f(x) - p_n(x)\|_\infty$ converges uniformly on $I$.


We will discuss this interpolation strategy in more detail later.