Set 10: Second Order Linear ODEs - Part 2

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Ordinary Differential Equations
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General Solution

The IVP

$$L[y] = y'' + py' + qy = 0, \quad y(t_0) = y_0, \quad y'(t_0) = y_0',$$

where p and q are continuous on an open interval $\mathcal{I}$ containing t_0, has a unique solution.

We want a general solution given in terms of two solutions y_1 and y_2, to the homogeneous ODE:

$$L[y_1] = y_1'' + py_1' + qy_1 = 0$$

$$L[y_2] = y_2'' + py_2' + qy_2 = 0$$

$$y = c_1 y_1 + c_2 y_2$$

General Solution

- Assume y_1 and y_2 are given.
- When is $y = c_1 y_1 + c_2 y_2$ a general solution to $L[y] = 0$?
- Specifically, given arbitrary initial conditions y_0 and y_0' when can constants c_1 and c_2 be found so the unique solution to the IVP on $\mathcal{I}$ is $y = c_1 y_1 + c_2 y_2$?
- If this can be done for any initial condition then all solutions to the IVP can be expressed as $y = c_1 y_1 + c_2 y_2$.

Linear Algebra Aside

Theorem 10.1. *The linear system of equations*

$$\begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix}$$

has a unique solution for any δ_1 *and* δ_2 *if and only if*

$$\gamma_{11}\gamma_{22} - \gamma_{12}\gamma_{21} \neq 0$$

Linear Algebra Aside

Corollary 10.2. *If*

$$\alpha = \gamma_{11}\gamma_{22} - \gamma_{12}\gamma_{21} \neq 0$$

then the unique solution to the linear system of equations

$$\begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix}$$

is given by

$$\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \frac{1}{\alpha} \begin{pmatrix} \gamma_{22} & -\gamma_{12} \\ -\gamma_{21} & \gamma_{11} \end{pmatrix} \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix}$$

Linear Algebra Aside

Corollary 10.3. *If*

$$\gamma_{11}\gamma_{22} - \gamma_{12}\gamma_{21} = 0$$

then there are values of δ_1 and δ_2 such that the linear system of equations

$$\begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix}$$

has no solution and has an infinite number of solutions.

Linear Algebra Aside

Unique solution.

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$$

$$\begin{pmatrix} -1 \\ 4 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 5 \end{pmatrix}$$

Linear Algebra Aside

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$$

does not have a unique solution since $2 - 2 = 0$.

It has no solution since $\forall \xi_1,\ \xi_2$

$$\begin{pmatrix} \xi_1 + 2\xi_2 \\ \xi_1 + 2\xi_2 \end{pmatrix} \neq \begin{pmatrix} 3 \\ 5 \end{pmatrix}$$

Linear Algebra Aside

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

does not have a unique solution since $2 - 2 = 0$.

It has infinite number of solutions since $\forall \gamma$

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 - 2\gamma \\ 1 + \gamma \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} -2\gamma \\ \gamma \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

General Solution

To find c_1 and c_2 given solutions y_1 and y_2, and initial conditions $y(t_0) = y_0$ and $y'(t_0) = y'_0$, we must solve

$$\begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y'_1(t_0) & y'_2(t_0) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} y_0 \\ y'_0 \end{pmatrix}$$

This has a unique solution for all $(y_0,\ y'_0)$ if and only if

$$y_1(t_0)y'_2(t_0) - y_2(t_0)y'_1(t_0) \neq 0$$

Solvability of IVP

A formal statement of this simple fact is given by:

Theorem 10.4 (Textbook page 149)**.** *The IVP*

$$L[y] = y'' + py' + qy = 0, \quad y(t_0) = y_0, \quad y'(t_0),$$

where p and q are continuous on an open interval $\mathcal{I}$ containing t_0, has a unique solution of the form

$$y = c_1 y_1 + c_2 y_2$$

where $L[y_1] = 0$ and $L[y_2] = 0$ if and only if

$$W(y_1, y_2)(t_0) = y_1(t_0)y_2'(t_0) - y_2(t_0)y_1'(t_0) \neq 0.$$

($W(y_1, y_2)(t)$ is called the Wronskian of y_1 and y_2 and is a function of t.)

Solvability of IVP

A fundamental theorem of linear second order ODEs can now be stated:

Theorem 10.5 (Textbook page 149)**.** *If*

$$L[y] = y'' + py' + qy = 0, \quad L[y_1] = 0, \quad L[y_2] = 0$$

where p and q are continuous on an open interval $\mathcal{I}$ then the family of functions

$$y = c_1 y_1 + c_2 y_2$$

where c_1 and c_2 are arbitrary constants contains all solutions of the ODE

$$L[y] = 0$$

if and only if

$$\exists t_0 \in \mathcal{I} \quad \textit{such that} \quad W(y_1, y_2)(t_0) \neq 0$$

Solvability of IVP

- Theorem 10.5 says that y_1 and y_2 define a general solution of $L[y] = 0$ if and only if $W(y_1, y_2)(t)$ is not 0 everywhere on $\mathcal{I}$.
- Theorem 10.5 follows from the existence and uniquness theorem presented earlier and Theorem 10.4 which gives the form of the unique solution. Since all possible solutions can be characterized by a choice of initial conditions, $y = c_1 y_1 + c_2 y_2$ must give all possible solutions.
- y_1 and y_2 are called a fundamental set of solutions of $L[y] = 0$ if $W(y_1, y_2)(t)$ is not 0 everywhere on $\mathcal{I}$.

Example

$$L[y] = y'' - y$$

$$y_1 = e^t, \quad y_2 = e^{-t}$$

$$y_1' = e^t, \quad y_2' = -e^{-t}$$

$$W(y_1, y_2)(t) = y_1 y_2' - y_2 y_1'$$

$$= -e^{-t}e^t - e^{-t}e^t = -1 - 1 = -2 \neq 0$$

- $\{y_1, \ \ y_2\}$ is a fundamental set of solutions
- $y = c_1 y_1 + c_2 y_2$ is a general solution of $L[y] = 0$

Example

$$L[y] = y'' + 9y$$

$$y_1 = \cos 3t, \quad y_2 = \sin 3t$$

$$y_1' = -3\sin 3t, \quad y_2' = 3\cos 3t$$

$$W(y_1, y_2)(t) = y_1 y_2' - y_2 y_1'$$

$$= 3\cos^2 3t + 3\sin^2 3t = 3 \neq 0$$

- $\{y_1, \ \ y_2\}$ is a fundamental set of solutions
- $y = c_1 y_1 + c_2 y_2$ is a general solution of $L[y] = 0$

Example

$$L[y] = y'' + 4y' + 4y$$

$$y_1 = e^{-2t}, \quad y_2 = te^{-2t}$$

$$y_1' = -2e^{-2t}, \quad y_2' = -2te^{-2t} + e^{-2t}$$

$$W(y_1, y_2)(t) = y_1 y_2' - y_2 y_1'$$

$$= e^{-2t}\left(e^{-2t} - 2te^{-2t}\right) - (te^{-2t})(-2e^{-2t})$$

$$= e^{-4t}\left(1 - 2t\right) + e^{-4t}(2t) = e^{-4t} \neq 0$$

- $\{y_1, \ \ y_2\}$ is a fundamental set of solutions
- $y = c_1 y_1 + c_2 y_2$ is a general solution of $L[y] = 0$

Use of Theorem

The previous examples had specific solutions y_1 and y_2 for a specific $L[y]$. The Wronskian can be used to prove more general statments.

Theorem 10.6. *If* $r_1 \neq r_2$, $y_1 = e^{r_1 t}$ *and* $y_2 = e^{r_2 t}$ *then*

$$y = c_1 y_1 + c_2 y_2$$

is a general solution for any $L[y] = 0$ *such that* $L[y_1] = 0$ *and* $L[y_2] = 0$.

Proof.

$$\begin{aligned} W(e^{r_1 t}, e^{r_2 t})(t) &= r_2 e^{r_2 t} e^{r_1 t} - r_1 e^{r_1 t} e^{r_2 t} \\ &= (r_2 - r_1) e^{r_2 t} e^{r_1 t} \neq 0 \end{aligned}$$

□

Use of Theorem

Theorem 10.7. *If* $y_1 = e^{rt}$ *and* $y_2 = te^{rt}$ *then*

$$y = c_1 y_1 + c_2 y_2$$

is a general solution for any $L[y] = 0$ *such that* $L[y_1] = 0$ *and* $L[y_2] = 0$.

Proof.

$$W(e^{rt}, te^{rt})(t) = e^{rt}(e^{rt} + re^{rt}) - (te^{rt})(re^{rt})$$
$$= e^{2rt}(1 + rt) - e^{2rt}(rt) = e^{2rt} \neq 0$$

□

Existence of Fundamental Solutions

Theorem 10.8 (Textbook page 151)**.** *Consider the ODE*

$$L[y] = y'' + py' + qy = 0$$

where p *and* q *are continuous on an open interval* $\mathcal{I}$*. If* y_1 *and* y_2 *are such that*

$$L[y_1] = 0, \quad y_1(t_0) = 1, \quad y_1'(t_0) = 0$$
$$L[y_2] = 0, \quad y_2(t_0) = 0, \quad y_2'(t_0) = 1$$

then $\{y_1, \; y_2\}$ *is a fundamental set of solutions for* $L[y] = 0$*.*

Existence of Fundamental Solutions

- y_1 and y_2 always exist by the existence and uniqueness theorem.
- Still requires the solution of two IVPs, so it is just as difficult as the original ODE.
- Example: $\{y_1, y_2\} = \{\sinh t, \;\; \cosh t\}$ is a fundamental set of solutions for $y'' - y$. (see Textbook page 152)

Abel's Theorem

The Wronskian of pairs of solutions of $L[y] = y'' + py' + qy = 0$ has a simple form.

Theorem 10.9 (Textbook page 154)**.** *If*

$$L[y] = y'' + py' + qy = 0$$

where p and q are continuous on an open interval $\mathcal{I}$, $L[y_1] = 0$ and $L[y_2] = 0$ then the Wronskian

$$W(y_1,\ y_2)(t) = CG(t) = W(t)$$

$$G(t) = \exp\left[-\int p(t)dt\right]$$

and C is a constant with respect to t that depends on y_1 and y_2.

Abel's Theorem

- $\forall t \in \mathcal{I}, \quad G(t) \neq 0$
- Since C is a constant, the Wronskian $W(t)$ is either always 0 or always nonzero on $\mathcal{I}$. So we need only test at a convenient t_0.
- $W(t)$ is equal up to a multiplicative constant for all fundamental sets of solutions.
- $W(t)$ can be determined up to a multiplicative constant without solving the ODE.

Abel's Theorem

Recall our examples:

- For $L[y] = y'' - y$ and $L[y] = y'' + 9y$ we have

$$p(t) = 0 \longrightarrow G(t) = e^c \longrightarrow W(t) = C$$

  We had -2 and 3 for the solution pairs chosen.

- For $L[y] = y'' + 4y' + 4y$ we have

$$p(t) = 4 \longrightarrow G(t) = e^{-4t} \longrightarrow W(t) = Ce^{-4t}$$

  For our example of $\left\{y_1,\ y_2\right\} = \left\{e^{-2t},\ te^{-2t}\right\}$ we had $C = 1$.

Summary

- Wronskian: $W(y_1,\ y_2)(t) = y_1 y_2' - y_2 y_1'$
- $\{y_1,\ y_2\}$ is a fundamental set of solutions if you can find $t_0 \in \mathcal{I}$ where $W(y_1,\ y_2)(t_0) \neq 0$
- Given a fundamental set of solutions $y = c_1 y_1 + c_2 y_2$ is the general solution of $L[y]$.
- Solving $L[y_1] = 0, y(t_0) = 1, y'(t_0) = 0$ and $L[y_2] = 0, y(t_0) = 0, y'(t_0) = 1$ gives a fundamental set.
- Unique solution to IVP can be found by determining c_1 and c_2 so that y satisfies the initial conditions.
- $W(y_1,\ y_2)(t) = W(t) = C \exp\left[\int p dt\right]$ where C is independent of t.

Generating Solutions of $L[y] = 0$

Suppose you find y_1 such that $L[y_1] = 0$ but have no idea about y_2, e.g., with a repeated root. The reduction of order method can be applied to a nonconstant coefficient $L[y] = 0$.

Given $y_1(t)$ find $y_2(t) = v(t)y_1(t)$.

$$y_2 = vy_1, \quad y_2' = v'y_1 + vy_1'$$

$$y_2'' = v''y_1 + v'y_1' + v'y_1' + vy_1'' = v''y_1 + 2v'y_1' + vy_1''$$

$$y_2'' + py_2' + qy_2 = (v''y_1 + 2v'y_1' + vy_1'') + p(v'y_1 + vy_1') + vy_1 = 0$$

Generating Solutions of $L[y] = 0$

$$\begin{aligned} y_2'' + py_2' + qy_2 &= (v''y_1 + 2v'y_1' + vy_1'') + p(v'y_1 + vy_1') + vy_1 \\ &= (y_1)v'' + (2y_1' + py_1)v' + (y_1'' + py_1' + qy_1) \\ &= (y_1)v'' + (2y_1' + py_1)v' = 0 \end{aligned}$$

Generating Solutions of $L[y] = 0$

Let $w = v'$

$$(y_1)v'' + (2y_1' + py_1)v' = 0$$
$$(y_1)w' + (2y_1' + py_1)w = 0$$

- solve for $w = v'$ using separable or linear ODE method
- integrate to get v
- recover $y_2(t) = v(t)y_1(t)$

Example

$$y'' + 4y' + 4y = 0$$
$$y_1 = e^{-2t}, \quad y_2 = vy_1$$

$$(y_1)w' + (2y_1' + py_1)w = 0$$
$$(e^{-2t})w' + (-4e^{-2t} + 4e^{-2t})w = 0 \rightarrow w' = 0$$
$$w' = 0 \rightarrow w = C = v' \rightarrow v = Ct + D$$

$$y_2(t) = v(t)y_1(t) = te^{-2t}$$

which we verified earlier yields a fundamental set of solutions.