Set 11: Second Order Linear ODEs - Part 3

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Ordinary Differential Equations
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Nonhomogeneous Problems

The IVP

$$y'' + p(t)y' + q(t)y = g(t), \quad y(t_0) = y_0, \quad y'(t_0),$$

where p, q and g are continuous on an open interval $\mathcal{I}$ containing t_0, has a unique solution.

The problem is nonhomogeneous since $g(t) \neq 0$.

A solution to this problem is called a particular solution to the ODE.

Solution Form

See Theorems 3.5.1 and 3.5.2 in the textbook.

- If Y_1 and Y_2 are particular solutions of a nonhomogeneous ODE then $Y_1 - Y_2$ is a solution to the homogeneous form of the ODE, i.e.,

  $$Y_1 = Y_2 + c_1 y_1 + c_2 y_2$$

  So adding a homogenous solution to a particular solution gives another particular solution.
- The general solution of a nonhomgenous ODE has the form

  $$y = Y + c_1 y_1 + c_2 y_2$$

  where Y is a particular solution and $\{y_1, y_2\}$ is a fundamental set of solutions to the homogeneous form of the ODE.

Particular Solutions of Linear Order 2 ODEs

Two methods will be discussed:

- Method of Undetermined Coefficients
  - Given form of $g(t)$, assume a parameterized form of $y(t)$.
  - Determine value of parameters so that $L[y] = g$.
- Variation of Parameters
  - Given a fundamental set of solutions $\{y_1, y_2\}$, assume $y(t) = A_1(t)y_1(t) + A_2(t)y_2(t)$.
  - Solve ODEs that determine $A_1(t)$ and $A_2(t)$ such that $L[y] = g$
  - Closed form known.

Method of Undetermined Coefficients

$$\text{If} \quad g(t) = (a_0t^n + \cdots + a_{n-1}t + a_n)$$
$$\text{then} \quad Y(t) = t^s(A_0t^n + \cdots + A_{n-1}t + A_n)$$

$s = 0, 1, 2$ is taken as the smallest integer so that no term in $Y(t)$ is a solution to the homogeneous problem.

Method of Undetermined Coefficients

$$\text{If} \quad g(t) = e^{\alpha t}(a_0t^n + \cdots + a_{n-1}t + a_n)$$
$$\text{then} \quad Y(t) = e^{\alpha t}t^s(A_0t^n + \cdots + A_{n-1}t + A_n)$$

$s = 0, 1, 2$ is taken as the smallest integer so that no term in $Y(t)$ is a solution to the homogeneous problem.

Note that the polynomial can be taken as 1 to handle the case of $g(t) = e^{\alpha t}$.

Method of Undetermined Coefficients

$$\text{If} \quad g(t) = e^{\alpha t}(a_0t^n + \cdots + a_{n-1}t + a_n)(c\sin\beta t + d\cos\beta t)$$
$$\text{then} \quad Y(t) = t^s e^{\alpha t}(A_0t^n + \cdots + A_{n-1}t + A_n)\sin\beta t$$
$$+ t^s e^{\alpha t}(B_0t^n + \cdots + B_{n-1}t + B_n)\cos\beta t$$

$s = 0, 1, 2$ is taken as the smallest integer so that no term in $Y(t)$ is a solution to the homogeneous problem.

Note that the polynomial can be taken as 1 and $\alpha = 0$ to handle the case of $g(t) = (c\sin\beta t + d\cos\beta t)$

Method of Undetermined Coefficients

If the ODE is of the form

$$L[Y] = g_1(t) + g_2(t) + \cdots + g_k(t)$$

where the $g_i(t)$ are in different categories then solve for k particular solutions

$$L[Y_i] = g_i(t), \quad 1 \le i \le k$$

and take

$$Y = Y_1 + \cdots + Y_k$$

as the particular solution of the original ODE.

Example

Textbook page 179

$$L[y] = y'' - 3y' - 4y = 2e^{-t} = g(t)$$

$$y'' - 3y' - 4y = 0$$
$$r^2 - 3r - 4 = 0 \to r_1 = 4, \quad r_2 = -1$$

fundamental set of solutions: $\{y_1, y_2\} = \{e^{4t}, e^{-t}\}$

Example

Textbook page 179

Since $g = 2e^{-t}$ and e^{-t} is a solution to the homogeneous equation and te^{-t} is not we have:

$$Y(t) = Ate^{-t}$$
$$Y'(t) = Ae^{-t} - Ate^{-t}$$
$$Y''(t) = -2Ae^{-t} + Ate^{-t}$$

Example

Textbook page 179

Substituting yields,

$$L[Y] = g(t)$$
$$-2Ae^{-t} + Ate^{-t} - 3Ae^{-t} + 3Ate^{-t} - 4Ate^{-t} = 2e^{-t}$$
$$e^{-t}[-2A - 3A] + te^{-t}[A + 3A - 4A] = 2e^{-t}$$
$$e^{-t}[-5A] = 2e^{-t} \to A = -\frac{2}{5}$$
$$Y(t) = -\frac{2}{5}te^{-t}$$

Example

Textbook page 179

$$y'' - 3y' - 4y = 2e^{-t} = g(t)$$

fundamental set of solutions: $\{y_1, y_2\} = \{e^{4t}, e^{-t}\}$

a particular solution: $Y(t) = -\frac{2}{5}te^{-t}$

general solution: $y = c_1e^{4t} + c_2e^{-t} - \frac{2}{5}te^{-t}$

Variation of Parameters

Theorem 11.1 (Textbook page 188). *If p, q, and g are continuous functions on an open interval $\mathcal{I} : \alpha < t < \beta$, and if $\{y_1, y_2\}$ is a fundamental set of solutions of the homogeneous problem corresponding to the nonhomogeneous ODE*

$$y'' + py' + qy = g$$

then a particular solution is given by

$$Y(t) = -y_1(t) \int_{t_0}^{t} \frac{y_2(s)g(s)}{W(y_1,y_2)(s)}\, ds + y_2(t) \int_{t_0}^{t} \frac{y_1(s)g(s)}{W(y_1,y_2)(s)}\, ds$$

for any convenient $t_0 \in \mathcal{I}$.

Variation of Parameters

- Theorem 11.1 is more general than constant coefficients.
- Requires a fundamental set of solutions.
- Requires integration involving $g(s)$.
- Useful when studying how y varies with changes to $g(t)$.

Example

Textbook page 179

$$L[y] = y'' - 3y' - 4y = 2e^{-t} = g(t)$$
$$\{y_1, y_2\} = \{e^{4t}, e^{-t}\}$$

$$W(e^{4s}, e^{-s})(s) = -e^{4s}e^{-s} - 4e^{-s}e^{4s} = -5e^{3s}$$

Example

Textbook page 179

To apply the theorem we need:

$$A_1(t) = -\int_{t_0}^{t} \frac{y_2(s)g(s)}{W(y_1,y_2)(s)}\, ds$$
$$= \frac{1}{5}\int_{t_0}^{t} (e^{-3s})(e^{-s})(2e^{-s})\, ds = \int_{t_0}^{t} \frac{2(e^{-5s})}{5}\, ds$$

$$\frac{2}{25}\left[-e^{-5s}\right]_{t_0}^{t} = \frac{2}{25}e^{-5t_0} - \frac{2}{25}e^{-5t}$$

Example

Textbook page 179

To apply the theorem we also need:

$$A_2(t) = \int_{t_0}^{t} \frac{y_1(s)g(s)}{W(y_1,y_2)(s)}\,ds$$
$$= -\frac{1}{5}\int_{t_0}^{t}(e^{-3s})(e^{4s})(2e^{-s})\,ds = -\frac{2}{5}\int_{t_0}^{t} ds$$
$$-\frac{2}{5}\left[s\right]_{t_0}^{t} = \frac{2}{5}t_0 - \frac{2}{5}t$$

Example

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Combining we have

$$Y(t) = \frac{2}{25}(e^{-5t_0} - e^{-5t})y_1 + (\frac{2}{5}t_0 - \frac{2}{5}t)y_2$$
$$= \frac{2}{25}(e^{-5t_0} - e^{-5t})e^{4t} + (\frac{2}{5}t_0 - \frac{2}{5}t)e^{-t}$$
$$= \frac{2}{25}e^{-5t_0}e^{4t} - \frac{2}{25}e^{-5t}e^{4t} + \frac{2}{5}t_0e^{-t} - \frac{2}{5}te^{-t}$$
$$= (\frac{2}{25}e^{-5t_0})e^{4t} + (\frac{2}{5}t_0 - \frac{2}{25})e^{-t} - \frac{2}{5}te^{-t}$$

Note the first two terms form a homogeneous solution. So we differ from the undetermined coefficient method's particular solution by a homogeneous solution which is allowed by the theory.

Example

Textbook page 179

The general solution of the nonhomogeneous ODE is

$$y = c_1e^{4t} + c_2e^{-t} - \frac{2}{5}te^{-t}$$

This comes from either:

- adding the general homogeneous solution to the particular solution from the method of undetermined coefficients
- or adding the general homogeneous solution to the particular solution from the method of variation of parameters and then absorbing the first two terms of the particular solution into the homogeneous general solution