Set 12: Second Order Linear ODEs - Part 4

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Ordinary Differential Equations
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Particular Solutions of Linear Order 2 ODEs

Two methods have been presented:

- Method of Undetermined Coefficients
- Variation of Parameters

We now consider two examples:

$$y'' - 3y' - 4y = 2\sin t$$

$$y'' - 3y' - 4y = -8e^t \cos 2t$$

and derive their general solutions with both of the methods.

Example

$$y'' - 3y' - 4y = 2\sin t$$

$$r^2 - 3r - 4 = 0 \rightarrow r_1 = 4, \quad r_2 = -1$$

$$\{y_1, y_2\} = \{e^{4t}, e^{-t}\} \quad \text{fundamental set}$$

For Undetermined Coefficients $g = 2\sin t$ is not a solution to the homogeneous ODE, therefore

$$Y = A\sin t + B\cos t$$

Example

$$y'' - 3y' - 4y = 2\sin t$$

$$\{y_1, y_2\} = \{e^{4t}, e^{-t}\} \quad \text{fundamental set}$$

$$Y = A\sin t + B\cos t, \quad Y' = A\cos t - B\sin t, \quad Y'' = -A\sin t - B\cos t$$

$$Y'' - 3Y' - 4Y = 2\sin t$$

$$-A\sin t - B\cos t - 3A\cos t + 3B\sin t - 4A\sin t - 4B\cos t = 2\sin t$$

$$\sin t[-5A + 3B] + \cos t[-3A - 5B] = 2\sin t$$

$$-5A + 3B = 2 \quad \text{and} \quad -3A - 5B = 0$$

$$A = -\frac{5}{17}, \quad B = \frac{3}{17}$$

Example

$$y'' - 3y' - 4y = 2\sin t$$

$$\{y_1, y_2\} = \{e^{4t}, e^{-t}\} \quad \text{fundamental set}$$

$$Y = A\sin t + B\cos t = -\frac{5}{17}\sin t + \frac{3}{17}\cos t \quad \text{particular solution}$$

$$y = c_1 e^{4t} + c_2 e^{-t} - \frac{5}{17}\sin t + \frac{3}{17}\cos t \quad \text{general solution}$$

Example

$$y'' - 3y' - 4y = 2\sin t$$

$$r^2 - 3r - 4 = 0 \to r_1 = 4, \quad r_2 = -1$$

$$\{y_1, y_2\} = \{e^{4t}, e^{-t}\} \quad \text{fundamental set}$$

$$W(y_1, y_2)(t) = y_1 y_2' - y_2 y_1' = -e^{4t}e^{-t} - 4e^{-t}e^{4t} = -5e^{3t}$$

$$Y = A_1(t)y_1 + A_2(t)y_2$$

Example

$$A_1 = -\int_{t_0}^{t} \frac{y_2 g}{W(y_1, y_2)}\, ds$$

$$= \frac{1}{5}\int_{t_0}^{t} (e^{-3s})(e^{-s})(2\sin s)\, ds = \frac{2}{5}\int_{t_0}^{t} e^{-4s}\sin s\, ds$$

$$A_2 = \int_{t_0}^{t} \frac{y_1 g}{W(y_1, y_2)}\, ds$$

$$= -\frac{1}{5}\int_{t_0}^{t} (e^{-3s})(e^{4s})(2\sin s)\, ds = -\frac{2}{5}\int_{t_0}^{t} e^{s}\sin s\, ds$$

Example

$$\int e^{ax} \sin x \, dx = \frac{e^{ax}}{a^2+1}(a \sin x - \cos x)$$

$$A_1 = \frac{2}{5}\int_{t_0}^{t} e^{-4s} \sin s \, ds = \frac{2}{5}\frac{e^{-4t}}{17}(-4\sin t - \cos t) + \tilde{c}_1$$
$$= -\frac{2}{85}e^{-4t}(4\sin t + \cos t) + \tilde{c}_1$$

$$A_2 = -\frac{2}{5}\int_{t_0}^{t} e^{s} \sin s \, ds = -\frac{2}{5}\frac{e^{t}}{2}(\sin t - \cos t) + \tilde{c}_2$$
$$= -\frac{2}{10}e^{t}(\sin t - \cos t) + \tilde{c}_2$$

Example

$$
\begin{aligned}
Y &= A_1(t)y_1 + A_2(t)y_2 \\
&= -\frac{2}{85}e^{-4t}(4\sin t + \cos t)e^{4t} + \tilde{c}_1 e^{4t} - \frac{2}{10}e^{t}(\sin t - \cos t)e^{-t} + \tilde{c}_2 e^{-t} \\
&= (-\frac{8}{85} - \frac{2}{10})\sin t + (-\frac{2}{85} + \frac{2}{10})\cos t + \tilde{c}_1 e^{4t} + \tilde{c}_2 e^{-t} \\
&= -\frac{5}{17}\sin t + \frac{3}{17}\cos t + \tilde{c}_1 e^{4t} + \tilde{c}_2 e^{-t}
\end{aligned}
$$

Y is a particular solution, and note the part that can be absorbed into a homogeneous solution.

Example

$$y'' - 3y' - 4y = 2\sin t$$

$$\{y_1, y_2\} = \{e^{4t}, e^{-t}\} \quad \text{fundamental set}$$

$$\therefore Y = -\frac{5}{17}\sin t + \frac{3}{17}\cos t \quad \text{is a particular solution}$$

$$y = -\frac{5}{17}\sin t + \frac{3}{17}\cos t + c_1 e^{4t} + c_2 e^{-t} \quad \text{is the general solution}$$

which is the same as derived using Undetermined Coefficients.

Example

$$y'' - 3y' - 4y = -8e^t \cos 2t$$

$$r^2 - 3r - 3r - 4 = 0 \to r_1 = 4, \quad r_2 = -1$$

$$\{y_1, y_2\} = \{e^{4t}, e^{-t}\} \quad \text{fundamental set}$$

For Undetermined Coefficients $g = -8e^t \cos 2t$ is not a solution to the homogeneous ODE. Therefore,

$$Y = e^t (A \cos 2t + B \sin 2t)$$

Example

$$y'' - 3y' - 4y = -8e^t \cos 2t$$

$$Y = e^t(A \cos 2t + B \sin 2t)$$

$$Y' = e^t(A + 2B) \cos 2t + e^t(B - 2A) \sin 2t$$

$$Y'' = e^t(4B - 3A) \cos 2t - e^t(4A + 3B) \sin 2t$$

$$Y'' - 3Y' - 4Y = -8e^t \cos 2t$$

$$-8e^t \cos 2t = e^t \cos 2t[4B - 3A - 3A - 6B - 4A]$$

$$+e^t \sin 2t[-4A - 3B - 3B + 6A - 4B]$$

$$e^t \cos 2t[-10A - 2B] + e^t \sin 2t[2A - 10B] = -8e^t \cos 2t$$

Example

$$e^t \cos 2t[-10A - 2B] + e^t \sin 2t[2A - 10B] = -8e^t \cos 2t$$

$$\begin{pmatrix} -10 & -2 \\ 2 & -10 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} -8 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} A \\ B \end{pmatrix} = \frac{1}{104} \begin{pmatrix} -10 & 2 \\ -2 & -10 \end{pmatrix} \begin{pmatrix} -8 \\ 0 \end{pmatrix} = \frac{1}{104} \begin{pmatrix} 80 \\ 16 \end{pmatrix} = \begin{pmatrix} \frac{10}{13} \\ \frac{2}{13} \end{pmatrix}$$

Example

$$y'' - 3y' - 4y = -8e^t \cos 2t$$

$$\{y_1, y_2\} = \{e^{4t}, e^{-t}\} \quad \text{fundamental set}$$

$$Y = \frac{10}{13}e^t \cos 2t + \frac{2}{13}e^t \sin 2t \quad \text{is a particular solution}$$

$$y = c_1 e^{4t} + c_2 e^{-t} + \frac{10}{13}e^t \cos 2t + \frac{2}{13}e^t \sin 2t \quad \text{is the general solution}$$

Example

$$y'' - 3y' - 4y = -8e^t \cos 2t$$

$$r^2 - 3r - 4 = 0 \rightarrow r_1 = 4, \quad r_2 = -1$$

$$\{y_1, y_2\} = \{e^{4t}, e^{-t}\} \quad \text{fundamental set}$$

$$W(y_1, y_2)(t) = y_1 y_2' - y_2 y_1' = -e^{4t}e^{-t} - 4e^{-t}e^{4t} = -5e^{3t}$$

$$Y = A_1(t)y_1 + A_2(t)y_2$$

Example

$$A_1 = -\int_{t_0}^{t} \frac{y_2 g}{W(y_1, y_2)}\, ds$$

$$= \frac{1}{5}\int_{t_0}^{t} (e^{-3s})(e^{-s})(-8e^{s}\cos 2s)\, ds = -\frac{8}{5}\int_{t_0}^{t} e^{-3s}\cos 2s\, ds$$

$$A_2 = \int_{t_0}^{t} \frac{y_1 g}{W(y_1, y_2)}\, ds$$

$$= -\frac{1}{5}\int_{t_0}^{t} (e^{-3s})(e^{4s})(-8e^{s}\cos 2s)\, ds = \frac{8}{5}\int_{t_0}^{t} e^{2s}\cos 2s\, ds$$

Example

$$\int e^{ax} \sin nx \; dx = \frac{e^{ax}}{a^2 + n^2}(a \sin nx - n \cos nx)$$

$$\int e^{ax} \cos nx \; dx = \frac{e^{ax}}{a^2 + n^2}(a \cos nx + n \sin nx)$$

$$A_1 = -\frac{8}{5} \int_{t_0}^{t} e^{-3s} \cos 2s \; ds$$

$$= -\frac{8}{5} \frac{e^{-3t}}{13}(-3 \cos 2t + 2 \sin 2t) + \tilde{c}_1$$

$$= \frac{24}{65} e^{-3t} \cos 2t - \frac{16}{65} e^{-3t} \sin 2t + \tilde{c}_1$$

Example

$$\int e^{ax} \sin nx \; dx = \frac{e^{ax}}{a^2 + n^2}(a \sin nx - n \cos nx)$$

$$\int e^{ax} \cos nx \; dx = \frac{e^{ax}}{a^2 + n^2}(a \cos nx + n \sin nx)$$

$$A_2 = \frac{8}{5} \int_{t_0}^{t} e^{2s} \cos 2s \; ds$$

$$= \frac{8}{5} \frac{e^{2t}}{8} [2 \cos 2t + 2 \sin 2t] + \tilde{c}_2 = \frac{2}{5} e^{2t} \cos 2t + \frac{2}{5} e^{2t} \sin 2t + \tilde{c}_2$$

Example

So removing the parts that can be absorbed into the homogeneous solutions:

$$\begin{aligned} Y &= (\frac{24}{65}e^{-3t}\cos 2t - \frac{16}{65}e^{-3t}\sin 2t)e^{4t} + (\frac{2}{5}e^{2t}\cos 2t + \frac{2}{5}e^{2t}\sin 2t)e^{-t} \\ &= e^t \cos 2t\,[\frac{24}{65} + \frac{2}{5}] + e^t \sin 2t\,[\frac{2}{5} - \frac{16}{65}] \\ &= \frac{10}{13}e^t \cos 2t + \frac{2}{13}e^t \sin 2t \end{aligned}$$

is a particular solution.

Example

The general solution derived via Variation of Parameters is therefore

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 + Y \\ &= c_1 e^{4t} + c_2 e^{-t} + \frac{10}{13} e^t \cos 2t + \frac{2}{13} e^t \sin 2t \end{aligned}$$

which is the same as that derived via Undetermined Coefficients.