Set 13: Vibrations Part 1

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Ordinary Differential Equations

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Mechanical and Electrical Vibration

$$au'' + bu' + cu = F(t)$$

$$a > 0, \quad c > 0, \quad b \geq 0$$

- simple model of
  - mass hanging on a vertical spring
  - mass sliding on a surface connected to a horizontal spring
  - current in a simple RLC circuit
- generalizations to linear systems for structures and circuits

$$Mu'' + Du' + Ku = F(t)$$

$$M, \; D, \; K \in \mathbb{R}^{n \times n}$$

- generalizations to systems with nonlinear elements, e.g., transistors

Mechanical Vibration

- mass hanging on a vertical spring
- $F = ma$ Newton and $F_s = -kL$ Hooke's Law for a spring
- unweighted length ℓ
- equilibrium position yields length of spring $\ell + L > 0$

$$mg = kL, \quad m > 0, \;\; k > 0, \;\; g > 0$$

- $u(t)$ represents displacement from equilibrium where
  - spring is stretched/mass moved down $\rightarrow u > 0$
  - spring is compressed/mass moved up $\rightarrow u < 0$

Mechanical Vibration

$$mu'' = \text{forces}$$

- $mg > 0$ force of gravity stretching the spring
- $-k(L+u)$ force of spring resisting stretching and compression (note the sign change is as $L+u$ changes sign)
- $-\gamma u', \quad \gamma > 0$ force of damping slowing velocity (note the sign change as u' changes sign)
- damping from
  - friction of surface, guides, enclosing fluid, etc.
  - damping device: dashpot

Mechanical Vibration

$$mu'' = \text{forces}$$

$$mu'' = mg - \gamma u' - k(L + u)$$

$$mu'' = mg - kL - \gamma u' - ku$$

$$mu'' = -\gamma u' - ku$$

$$mu'' + \gamma u' + ku = 0$$

$$u(0) = \text{initial displacement}$$

$$u'(0) = \text{initial velocity}$$

If there are other forces varying with time then

$$mu'' + \gamma u' + ku = F(t)$$

Mechanical Vibration

Undamped ($\gamma = 0$) and free ($F(t) = 0$) vibrations satisfy

$$mu'' + ku = 0$$

$$mr^2 + k = 0 \to r_\pm = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m} = \pm i\sqrt{\frac{k}{m}}$$

$$u(t) = A\cos\omega_0 t + B\sin\omega_0 t$$

- steady sinusoidal oscillation
- ω_0 is called the natural frequency where

$$\omega_0^2 = \frac{k}{m}$$

- period of oscillation

$$T = \frac{2\pi}{\omega_0}$$

Mechanical Vibration

Undamped ($\gamma = 0$) and free ($F(t) = 0$) vibrations satisfy

$$mu'' + ku = 0 \to u(t) = A\cos\omega_0 t + B\sin\omega_0 t$$

$$\omega_0^2 = \frac{k}{m}$$

A and B set via initial conditions $u(0) = u_0$ and $u'(0) = v_0$

Mechanical Vibration

The amplitude/phase representation of the sinusoid is often used for analysis and characterization purposes:

$$u(t) = A\cos\omega_0 t + B\sin\omega_0 t = R\cos(\omega_0 t - \delta)$$

$$R\cos(\omega_0 t - \delta) = R\big[\cos(\omega_0 t)\cos(-\delta) - \sin(\omega_0 t)\sin(-\delta)\big]$$

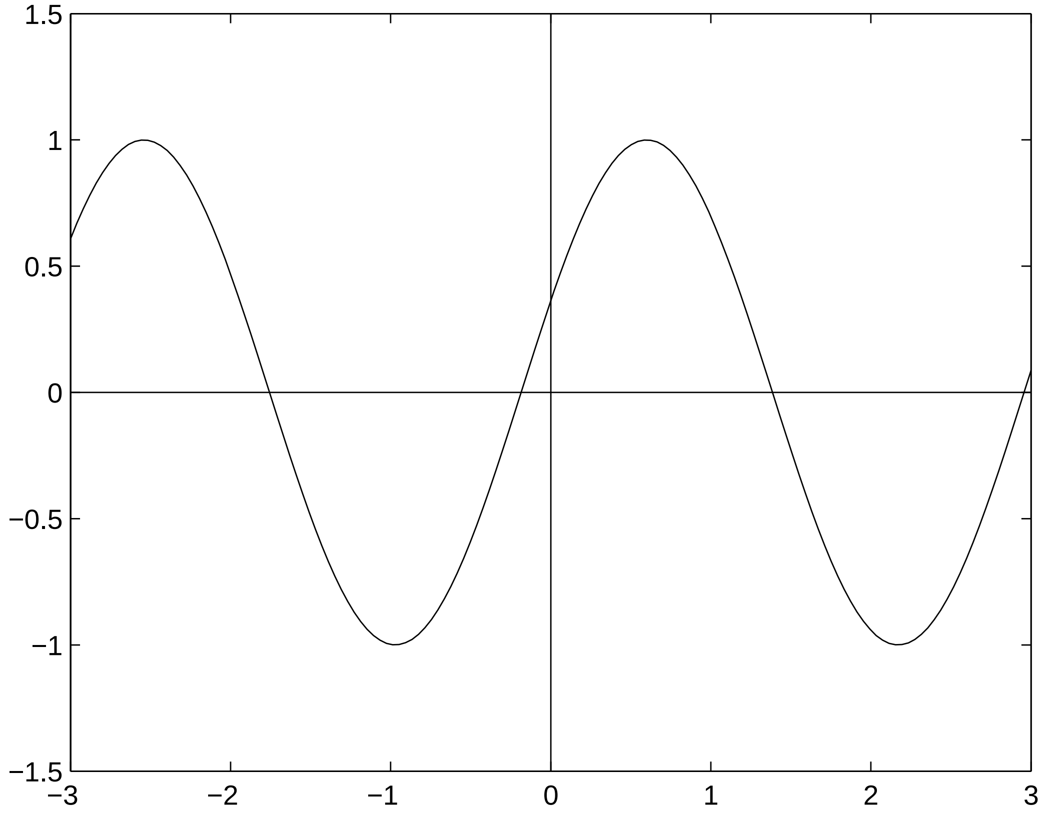
$$R\cos(\delta)\cos(\omega_0 t) + R\sin(\delta)\sin(\omega_0 t)$$

$$\therefore A = R\cos\delta, \quad B = R\sin\delta$$

$$A^2 + B^2 = R^2 \rightarrow R = +\sqrt{A^2 + B^2}$$

$$\tan\delta = \frac{B}{A} \rightarrow \delta = \arctan\frac{B}{A}$$

Amplitude/Phase Example



$$u(t) = \cos(2t - 1.2)$$

Mechanical Vibration

So given A and B we can recover R and δ.

We have:

- initial condition $u(0) = R\cos\delta$
- initial condition $u'(0) = R\omega_0 \sin\delta$
- $u(0) = R, \;\; u'(0) = 0 \rightarrow \delta = 0$
- $\omega_0 t = \delta + 2\pi j \rightarrow u(t) = R$
- the maximum magnitude is R
- the phase angle δ measures displacement from $\delta = 0$ to get initial condition

Note $\delta = \arctan\frac{B}{A}$ will have two solutions so use signs of A and B to choose.

Undamped and Free Vibration

$$mu'' + ku = 0$$

$$u(t) = A\cos\omega_0 t + B\sin\omega_0 t$$

$$u(t) = R\cos(\omega_0 t - \delta)$$

$$\omega_0^2 = \frac{k}{m} \quad \text{and} \quad T = \frac{2\pi}{\omega_0}$$

Assuming the model is appropriate:

- For fixed k, $m \to \infty$ implies $\omega \to 0$, slower oscillation
- For fixed m, $k \to \infty$ implies $\omega \to \infty$, faster oscillation
- maximum magnitude R is maintained for all time and achieved twice per period of oscillation
- no energy dissipation

Example

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$$mu'' + ku = 0$$

$$\frac{10}{32}u'' + 60u = 0 \rightarrow u'' + 192u = 0$$

$$\omega_0^2 = \frac{k}{m} = 192 = 64 \times 3 \rightarrow \omega_0 = 8\sqrt{3}$$

given the form of the equation we know the general solution is:

$$u(t) = A\cos(8\sqrt{3}t) + B\sin(8\sqrt{3}t)$$

$$T = \frac{2\pi}{8\sqrt{3}} \approx 0.45345$$

Example

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Impose initial conditions and note simple form of linear system when $t_0 = 0$:

$$u(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t)$$

$$u'(t) = -A\omega_0 \sin(\omega_0 t) + B\omega_0 \cos(\omega_0 t)$$

$$u(0) = \frac{1}{6}, \quad u'(0) = -1$$

$$\begin{pmatrix} 1 & 0 \\ 0 & \omega_0 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} \frac{1}{6} \\ -1 \end{pmatrix}$$

$$u(t) = \frac{1}{6}\cos(8\sqrt{3}t) - \frac{1}{8\sqrt{3}}\sin(8\sqrt{3}t)$$

Example

Textbook page 196

$$u(t) = \frac{1}{6}\cos(8\sqrt{3}t) - \frac{1}{8\sqrt{3}}\sin(8\sqrt{3}t) = R\cos(\omega_0 t - \delta)$$

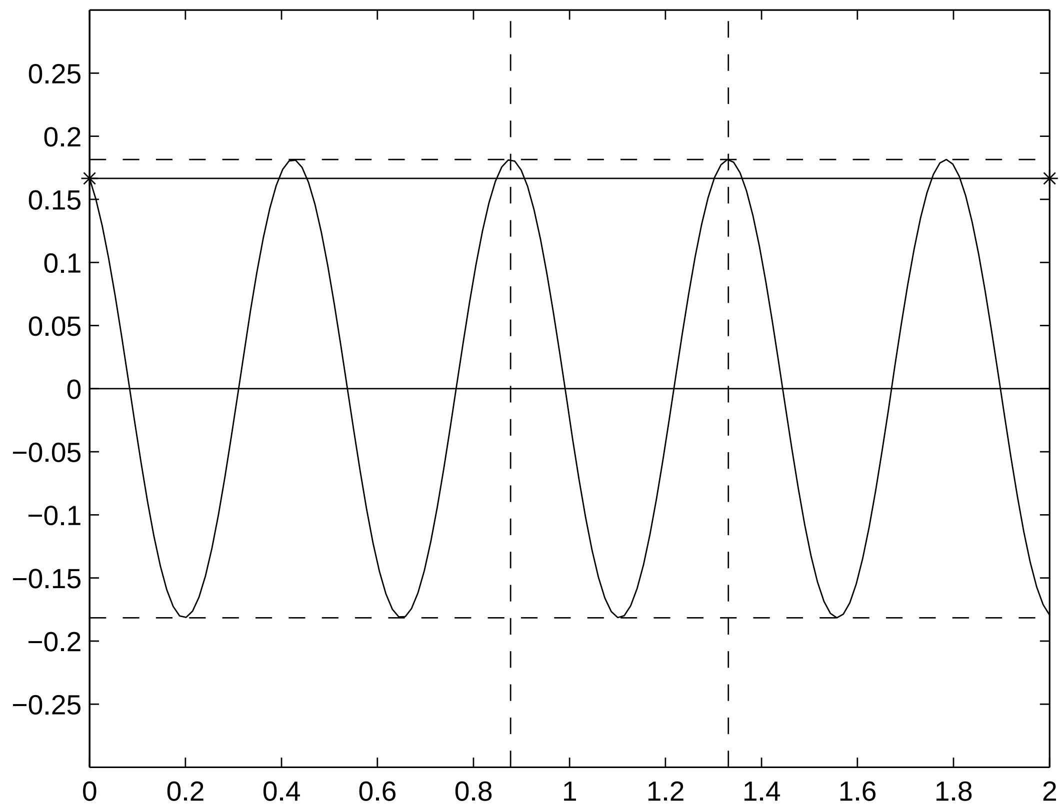
$$R^2 = \frac{1}{36} + \frac{1}{192} = \frac{19}{576} \to R \approx 0.18162$$

$$\tan\delta = -\frac{6}{8\sqrt{3}} = -\frac{\sqrt{3}}{4}$$

Since $A > 0$ and $B < 0$, we have $\cos\delta > 0$ and $\sin\delta < 0$

$$\delta = -arctan\frac{\sqrt{3}}{4} \approx -0.40864$$

Undamped and Free Vibration Example



$$u'' + 192u = 0,\ u(0) = \frac{1}{6},\ u'(0) = -1$$

Mechanical Vibration

Damped ($\gamma > 0$) and free ($F(t) = 0$) vibrations satisfy

$$mu'' + \gamma u' + ku = 0$$

$$mr^2 + k = 0 \to r_\pm = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m} = \frac{-\gamma}{2m} \pm \frac{\sqrt{\gamma^2 - 4mk}}{2m}$$

real part of conjugate pair always negative so amplitude decays

$$\frac{\gamma^2}{4km} < 1 \to u(t) = e^{-\gamma t/2m}\big(A\cos\mu t + B\sin\mu t\big)$$

$$\mu = \frac{\sqrt{4mk - \gamma^2}}{2m} > 0$$

Mechanical Vibration

- Not periodic but the time interval between successive maxima/minima/equilibria is constant.
- So the farther it has to move the faster it goes.

$$u(t) = e^{-\gamma t/2m}\big(A\cos\mu t + B\sin\mu t\big)$$

$$\mu = \frac{\sqrt{4mk-\gamma^2}}{2m} \quad \text{quasi-frequency,} \quad T_d = \frac{2\pi}{\mu} \quad \text{quasi-period}$$

$$\frac{\mu}{\omega_0} = \sqrt{1-\frac{\gamma^2}{4km}} \le 1, \quad \text{where} \quad \frac{\gamma^2}{4km} < 1$$

Damped and Free Mechanical Vibration

$$\frac{\gamma^2}{4km} < 1$$

- gives an idea of if the oscillation is lightly or heavily damped.
- gives an idea of how close the quasi-frequency is to the natural frequency.
- damping reduces the frequency compared to natural, i.e., "period" increases.
- Trends

$$\frac{\gamma^2}{4km} \to 0 \quad \text{then } \mu \to \omega_0$$
$$\frac{\gamma^2}{4km} \to 1 \quad \text{then } \mu \to 0?$$

Damped and Free Mechanical Vibration

What about the effect on amplitude?

$$u(t) = e^{-\gamma t/2m}\big(A\cos\mu t + B\sin\mu t\big)$$

$$A = R\cos\delta \quad \text{and} \quad B = R\sin\delta$$

$$u(t) = Re^{-\gamma t/2m}\cos(\mu t - \delta)$$

$$-Re^{-\gamma t/2m} \le u(t) \le Re^{-\gamma t/2m}$$

Damped and Free Mechanical Vibration

- $\pm Re^{-\gamma t/2m}$ is an envelope of the solution
- $\gamma/2m$ indicates how quickly decay occurs.
- Can use this to find τ such that $|u(t)| < D$ for all $t \geq \tau$

$$Re^{-\gamma t/2m} \leq D$$

  Solve for τ.

Damped and Free Vibration Example

Textbook page 199: Undamped and free vibration

$$u(0) = 2, \quad \text{and} \quad u'(0) = 0$$

$$u'' + u = 0$$

$$\omega_0 = 1 \quad \text{and} \quad T = 2\pi \approx 6.28318$$

$$u(t) = 2\cos t$$

Damped and Free Vibration Example

Textbook page 199: Damped and free vibration

$$u(0) = 2, \quad \text{and} \quad u'(0) = 0$$

$$u'' + 0.125u' + u = 0, \quad m = k = 1, \quad \gamma = 0.125 = 1/8$$

$$\mu = \frac{\sqrt{4 - \gamma^2}}{2} = \frac{1}{2}\sqrt{\frac{255}{64}} = \frac{\sqrt{255}}{16} \approx 0.998$$

$$\lambda = -\frac{\gamma}{2m} = -\frac{1}{16}, \quad T_d = \frac{2\pi}{\mu} \approx 6.295$$

$$u = e^{-t/16}\big(A\cos\frac{\sqrt{255}}{16}t + B\sin\frac{\sqrt{255}}{16}t\big)$$

Damped and Free Vibration Example

Textbook page 199: Damped and free vibration

$$u(0) = 2, \quad \text{and} \quad u'(0) = 0$$

$$u = e^{-t/16}\big(A\cos\frac{\sqrt{255}}{16}t + B\sin\frac{\sqrt{255}}{16}t\big)$$

$$A = 2, \quad B = \frac{2}{\sqrt{255}} \to R = \frac{32}{\sqrt{255}}$$

$$\tan\delta = \frac{1}{\sqrt{255}} \to \delta \approx 0.06254$$

$$u(t) = \frac{32}{\sqrt{255}}e^{-t/16}\cos\big(\frac{\sqrt{255}}{16}t - \delta\big)$$

Damped and Free Vibration Example

Textbook page 199

- light damping

$$\frac{\gamma^2}{4km} = \frac{1}{256} \approx 0.0039 < 1$$

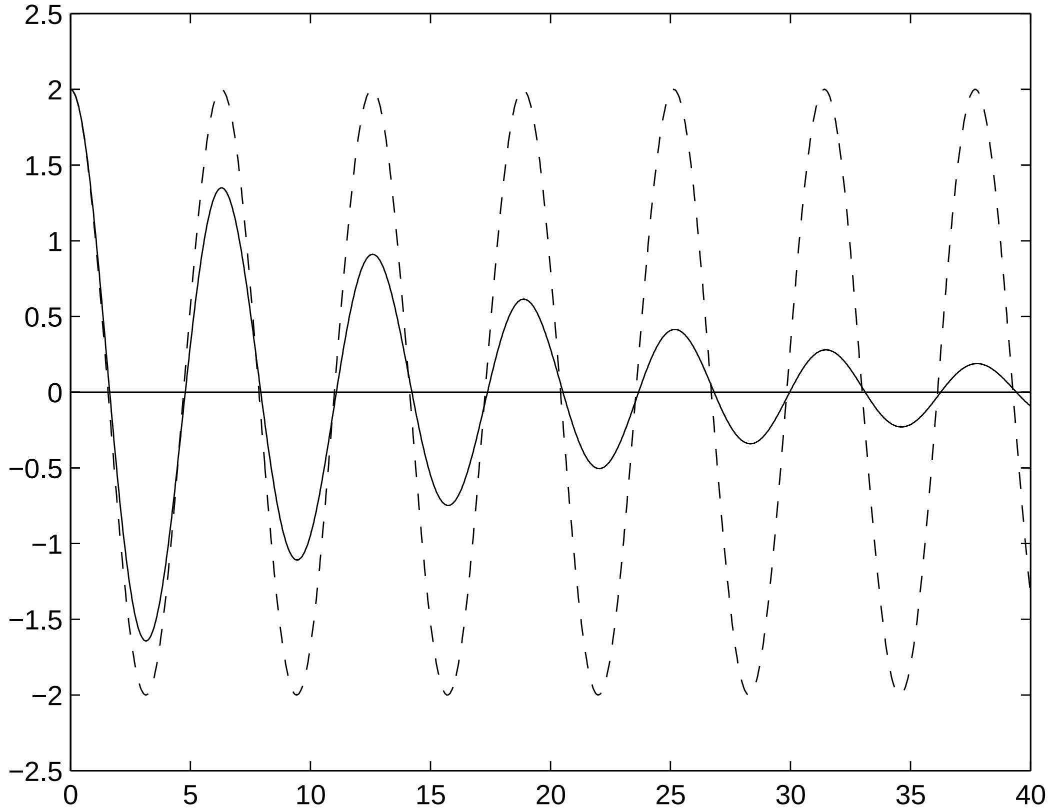
- quasi-period and natural period close

$$T_d \approx 6.295 \quad T \approx 6.28318$$

- amplitude damped noticeably

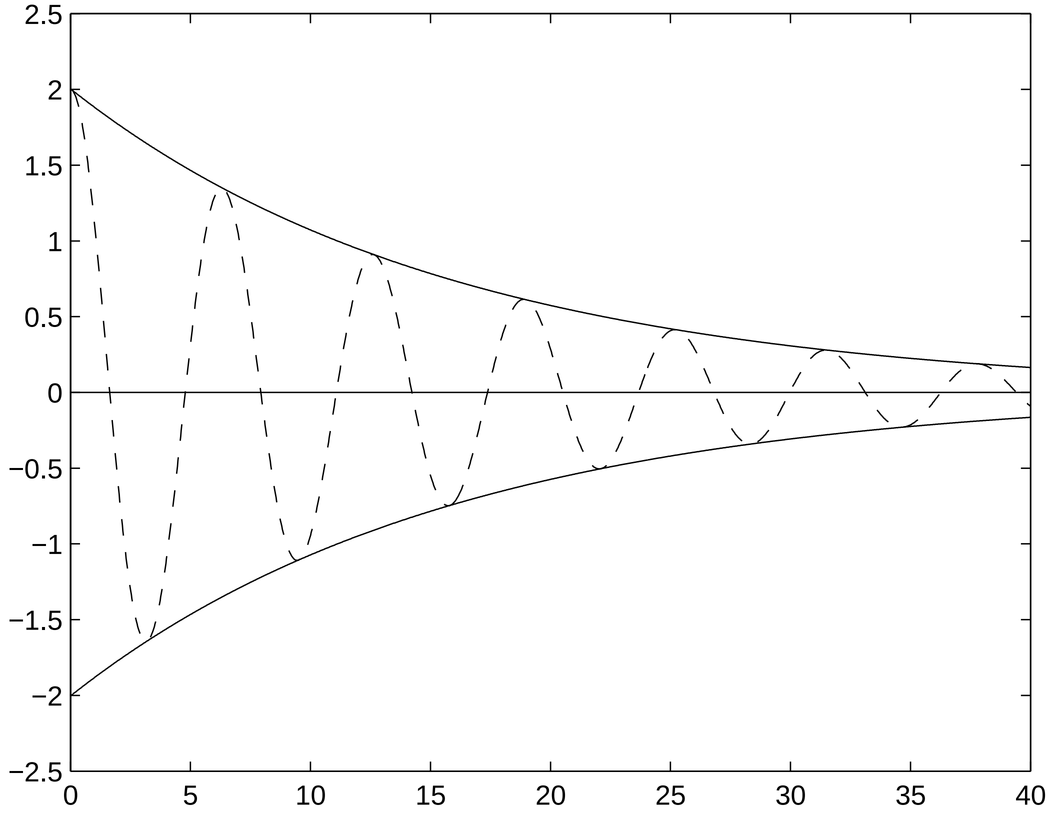
$$-\frac{\gamma}{2m} = -\frac{1}{16}$$

Damped and Free Vibration Example



$$u'' + 0.125u' + u = 0,\ u'' + u = 0,\ u(0) = 2,\ u'(0) = 0$$

Undamped and Free Vibration Example



$u'' + 0.125u' + u = 0$, $u(0) = 2$, $u'(0) = 0$, and envelope

Critical and Overdamped Free Vibration

- Damped oscillation

$$0 < \frac{\gamma^2}{4km} < 1$$

- Critical damping yields no oscillation only asymptotic convergence to equilibrium

$$\frac{\gamma^2}{4km} = 1 \rightarrow \gamma = \sqrt{4km}, \quad \mu = 0$$

- Overdamping when

$$\frac{\gamma^2}{4km} > 1 \quad \text{two distinct negative real roots}$$

Critically Damped Free Vibration

Textbook page 199

For undamped $u'' + u = 0$, $k = m = 1 \rightarrow \gamma = 2$ is critical damping

$$u'' + 2u' + u = 0$$

$$-\frac{\gamma}{2m} = -1$$

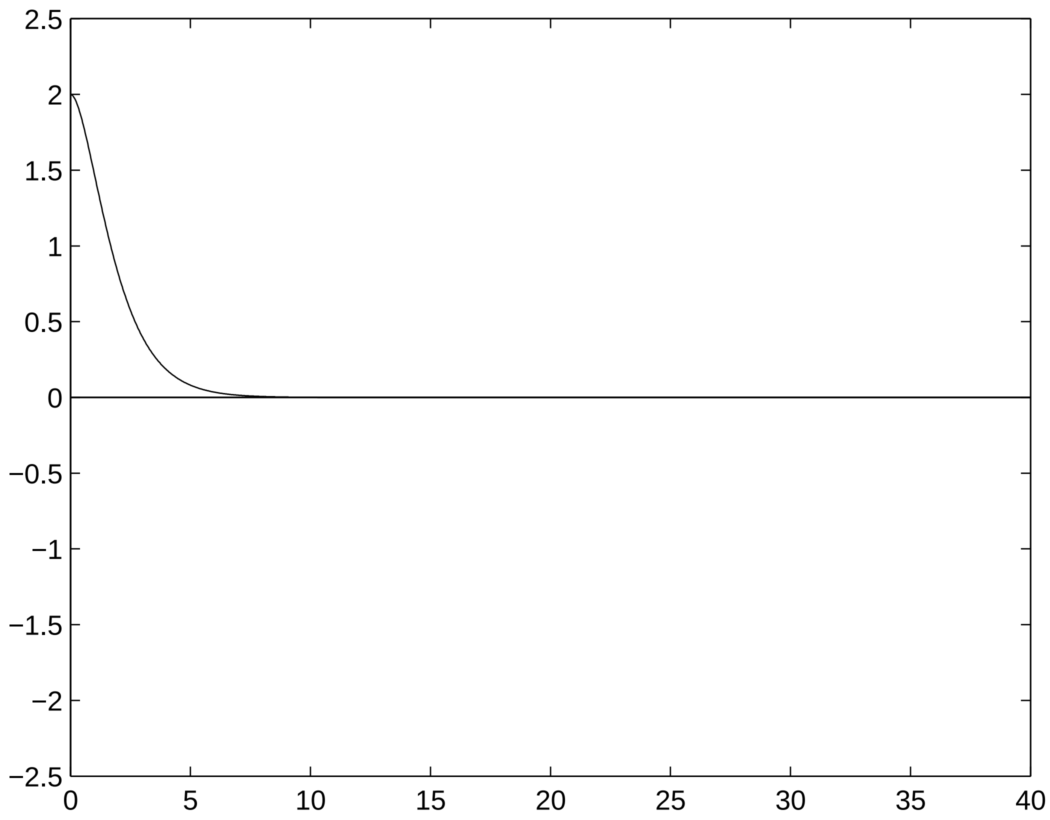
$$u(t) = e^{-t}(A + Bt)$$

$$u(0) = 2, \quad u'(0) = 0 \rightarrow A = B = 2$$

$$u(t) = 2e^{-t}(1 + t)$$

Comparatively very rapid damping toward equilibrium.

Critically Damped Free Vibration



$u'' + 2u' + u = 0$, $u(0) = 2$, $u'(0) = 0$, and $u = 2e^{-t}(1+t)$