

Set 13: Vibrations Part 1

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Ordinary Differential Equations
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Mechanical Vibration

- mass hanging on a vertical spring
- $F = ma$ Newton and $F_s = -kL$ Hooke's Law for a spring
- unweighted length ℓ
- equilibrium position yields length of spring $\ell + L > 0$
 $mg = kL$, $m > 0$, $k > 0$, $g > 0$
- $u(t)$ represents displacement from equilibrium where
 - spring is stretched/mass moved down $\rightarrow u > 0$
 - spring is compressed/mass moved up $\rightarrow u < 0$

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Mechanical and Electrical Vibration

$$au'' + bu' + cu = F(t)$$
$$a > 0, \quad c > 0, \quad b \geq 0$$

- simple model of
 - mass hanging on a vertical spring
 - mass sliding on a surface connected to a horizontal spring
 - current in a simple RLC circuit
- generalizations to linear systems for structures and circuits

$$Mu'' + Du' + Ku = F(t)$$

$$M, D, K \in \mathbb{R}^{n \times n}$$

- generalizations to systems with nonlinear elements, e.g., transistors

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Mechanical Vibration

$$mu'' = \text{forces}$$

- $mg > 0$ force of gravity stretching the spring
- $-k(L + u)$ force of spring resisting stretching and compression (note the sign change is as $L + u$ changes sign)
- $-\gamma u'$, $\gamma > 0$ force of damping slowing velocity (note the sign change as u' changes sign)
- damping from
 - friction of surface, guides, enclosing fluid, etc.
 - damping device: dashpot

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Mechanical Vibration

$mu'' = \text{forces}$

$$mu'' = mg - \gamma u' - k(L + u)$$

$$mu'' = mg - kL - \gamma u' - ku$$

$$mu'' = -\gamma u' - ku$$

$$mu'' + \gamma u' + ku = 0$$

$u(0) = \text{initial displacement}$

$u'(0) = \text{initial velocity}$

If there are other forces varying with time then

$$mu'' + \gamma u' + ku = F(t)$$

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Mechanical Vibration

Undamped ($\gamma = 0$) and free ($F(t) = 0$) vibrations satisfy

$$mu'' + ku = 0 \rightarrow u(t) = A \cos \omega_0 t + B \sin \omega_0 t$$

$$\omega_0^2 = \frac{k}{m}$$

A and B set via initial conditions $u(0) = u_0$ and $u'(0) = v_0$

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Mechanical Vibration

Undamped ($\gamma = 0$) and free ($F(t) = 0$) vibrations satisfy

$$mu'' + ku = 0$$

$$mr^2 + k = 0 \rightarrow r_{\pm} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m} = \pm i \sqrt{\frac{k}{m}}$$

$$u(t) = A \cos \omega_0 t + B \sin \omega_0 t$$

- steady sinusoidal oscillation

- ω_0 is called the natural frequency where

$$\omega_0^2 = \frac{k}{m}$$

- period of oscillation

$$T = \frac{2\pi}{\omega_0}$$

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Mechanical Vibration

The amplitude/phase representation of the sinusoid is often used for analysis and characterization purposes:

$$u(t) = A \cos \omega_0 t + B \sin \omega_0 t = R \cos(\omega_0 t - \delta)$$

$$R \cos(\omega_0 t - \delta) = R [\cos(\omega_0 t) \cos(-\delta) - \sin(\omega_0 t) \sin(-\delta)]$$

$$R \cos(\delta) \cos(\omega_0 t) + R \sin(\delta) \sin(\omega_0 t)$$

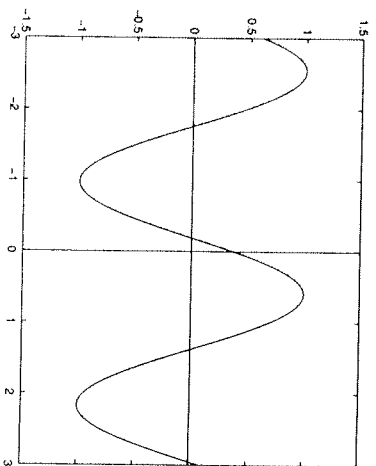
$$\therefore A = R \cos \delta, \quad B = R \sin \delta$$

$$A^2 + B^2 = R^2 \rightarrow R = +\sqrt{A^2 + B^2}$$

$$\tan \delta = \frac{B}{A} \rightarrow \delta = \arctan \frac{B}{A}$$

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Amplitude/Phase Example



$$u(t) = \cos(2t - 1.2)$$

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Mechanical Vibration

So given A and B we can recover R and δ .

We have:

- initial condition $u(0) = R \cos \delta$
- initial condition $u'(0) = R\omega_0 \sin \delta$
- $u(0) = R$, $u'(0) = 0 \rightarrow \delta = 0$
- $\omega_0 t = \delta + 2\pi j \rightarrow u(t) = R$
- the maximum magnitude is R
- the phase angle δ measures displacement from $\delta = 0$ to get initial condition

Note $\delta = \arctan \frac{B}{A}$ will have two solutions so use signs of A and B to choose.

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Undamped and Free Vibration

$$mu'' + ku = 0$$

$$u(t) = A \cos \omega_0 t + B \sin \omega_0 t$$

$$u(t) = R \cos(\omega_0 t - \delta)$$

$$\omega_0^2 = \frac{k}{m} \quad \text{and} \quad T = \frac{2\pi}{\omega_0}$$

Assuming the model is appropriate:

- For fixed k , $m \rightarrow \infty$ implies $\omega \rightarrow 0$, slower oscillation
- For fixed m , $k \rightarrow \infty$ implies $\omega \rightarrow \infty$, faster oscillation
- maximum magnitude R is maintained for all time and achieved twice per period of oscillation
- no energy dissipation

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Example

Textbook page 196

$$mu'' + ku = 0$$

$$\frac{10}{32}u'' + 60u = 0 \rightarrow u'' + 192u = 0$$

$$\omega_0^2 = \frac{k}{m} = 192 = 64 \times 3 \rightarrow \omega_0 = 8\sqrt{3}$$

given the form of the equation we know the general solution is:

$$u(t) = A \cos(8\sqrt{3}t) + B \sin(8\sqrt{3}t)$$

$$T = \frac{2\pi}{8\sqrt{3}} \approx 0.45345$$

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Example

Textbook page 196

Impose initial conditions and note simple form of linear system when $t_0 = 0$:

$$u(t) = A \cos(\omega_0 t) + B \sin(\omega_0 t)$$

$$u'(t) = -A\omega_0 \sin(\omega_0 t) + B\omega_0 \cos(\omega_0 t)$$

$$u(0) = \frac{1}{6}, \quad u'(0) = -1$$

$$\begin{pmatrix} 1 & 0 \\ 0 & \omega_0 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} \frac{1}{6} \\ -1 \end{pmatrix}$$

$$u(t) = \frac{1}{6} \cos(8\sqrt{3}t) - \frac{1}{8\sqrt{3}} \sin(8\sqrt{3}t)$$

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Example

Textbook page 196

$$u(t) = \frac{1}{6} \cos(8\sqrt{3}t) - \frac{1}{8\sqrt{3}} \sin(8\sqrt{3}t) = R \cos(\omega_0 t - \delta)$$

$$R^2 = \frac{1}{36} + \frac{1}{192} = \frac{19}{576} \rightarrow R \approx 0.18162$$

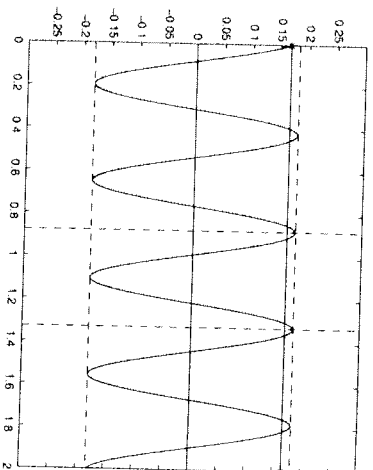
$$\tan \delta = -\frac{6}{8\sqrt{3}} = -\frac{\sqrt{3}}{4}$$

Since $A > 0$ and $B < 0$, we have $\cos \delta > 0$ and $\sin \delta < 0$

$$\delta = -\arctan \frac{\sqrt{3}}{4} \approx -0.40864$$

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Undamped and Free Vibration Example



$$u'' + 192u = 0, u(0) = \frac{1}{6}, u'(0) = -1$$

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Mechanical Vibration

Damped ($\gamma > 0$) and free ($F(t) = 0$) vibrations satisfy

$$mu'' + \gamma u' + ku = 0$$

$$mr^2 + k = 0 \rightarrow r_{\pm} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m} = \frac{-\gamma}{2m} \pm \frac{\sqrt{\gamma^2 - 4mk}}{2m}$$

real part of conjugate pair always negative so amplitude decays

$$\frac{\gamma^2}{4km} < 1 \rightarrow u(t) = e^{-\gamma t/2m} (A \cos \mu t + B \sin \mu t)$$

$$\mu = \frac{\sqrt{4mk - \gamma^2}}{2m} > 0$$

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Mechanical Vibration

- Not periodic but the time interval between successive maxima/minima/equilibria is constant.
- So the farther it has to move the faster it goes.

$$u(t) = e^{-\gamma t/2m} (A \cos \mu t + B \sin \mu t)$$

$$\mu = \frac{\sqrt{4mk - \gamma^2}}{2m} \quad \text{quasi-frequency, } T_d = \frac{2\pi}{\mu} \quad \text{quasi-period}$$

$$\frac{\mu}{\omega_0} = \sqrt{1 - \frac{\gamma^2}{4km}} \leq 1, \quad \text{where } \frac{\gamma^2}{4km} < 1$$

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Damped and Free Mechanical Vibration

- gives an idea of if the oscillation is lightly or heavily damped.
- gives an idea of how close the quasi-frequency is to the natural frequency.
- damping reduces the frequency compared to natural, i.e., "period" increases.
- Trends

$$\frac{\gamma^2}{4km} < 1$$

$$\frac{\gamma^2}{4km} \rightarrow 0 \quad \text{then } \mu \rightarrow \omega_0$$

$$\frac{\gamma^2}{4km} \rightarrow 1 \quad \text{then } \mu \rightarrow 0?$$

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Damped and Free Mechanical Vibration

What about the effect on amplitude?

$$u(t) = e^{-\gamma t/2m} (A \cos \mu t + B \sin \mu t)$$

$$A = R \cos \delta \quad \text{and} \quad B = R \sin \delta$$

$$u(t) = R e^{-\gamma t/2m} \cos(\mu t - \delta)$$

$$-R e^{-\gamma t/2m} \leq u(t) \leq R e^{-\gamma t/2m}$$

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Damped and Free Mechanical Vibration

- $\pm R e^{-\gamma t/2m}$ is an envelope of the solution
- $\gamma/2m$ indicates how quickly decay occurs.
- Can use this to find τ such that $|u(t)| < D$ for all $t \geq \tau$

$$R e^{-\gamma t/2m} \leq D$$

Solve for τ .

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Damped and Free Vibration Example

Textbook page 199: Undamped and free vibration

$$\begin{aligned}u(0) &= 2, \text{ and } u'(0) = 0 \\u'' + u &= 0 \\ \omega_0 &= 1 \text{ and } T = 2\pi \approx 6.28318 \\ u(t) &= 2 \cos t\end{aligned}$$

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Damped and Free Vibration Example

Textbook page 199: Damped and free vibration

$$\begin{aligned}u(0) &= 2, \text{ and } u'(0) = 0 \\u'' + 0.125u' + u &= 0, \quad m = k = 1, \quad \gamma = 0.125 = 1/8 \\ \mu &= \frac{\sqrt{4 - \gamma^2}}{2} = \frac{1}{2} \sqrt{\frac{255}{64}} = \frac{\sqrt{255}}{16} \approx 0.998 \\ \lambda &= -\frac{\gamma}{2m} = -\frac{1}{16}, \quad T_d = \frac{2\pi}{\mu} \approx 6.295 \\ u &= e^{-t/16} \left(A \cos \frac{\sqrt{255}}{16} t + B \sin \frac{\sqrt{255}}{16} t \right)\end{aligned}$$

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Damped and Free Vibration Example

Textbook page 199: Damped and free vibration

$$\begin{aligned}u(0) &= 2, \text{ and } u'(0) = 0 \\ u &= e^{-t/16} \left(A \cos \frac{\sqrt{255}}{16} t + B \sin \frac{\sqrt{255}}{16} t \right) \\ A &= 2, \quad B = \frac{2}{\sqrt{255}} \rightarrow R = \frac{32}{\sqrt{255}} \\ \tan \delta &= \frac{1}{\sqrt{255}} \rightarrow \delta \approx 0.06254 \\ u(t) &= \frac{32}{\sqrt{255}} e^{-t/16} \cos \left(\frac{\sqrt{255}}{16} t - \delta \right)\end{aligned}$$

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Damped and Free Vibration Example

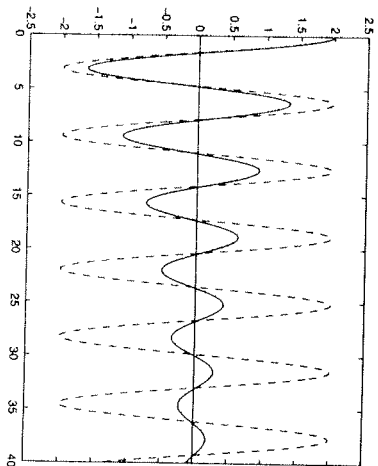
Textbook page 199

- light damping
- quasi-period and natural period close
- amplitude damped noticeably

$$\begin{aligned}\frac{\gamma^2}{4km} &= \frac{1}{256} \approx 0.0039 < 1 \\ T_d &\approx 6.295 \quad T \approx 6.28318 \\ -\frac{\gamma}{2m} &= -\frac{1}{16}\end{aligned}$$

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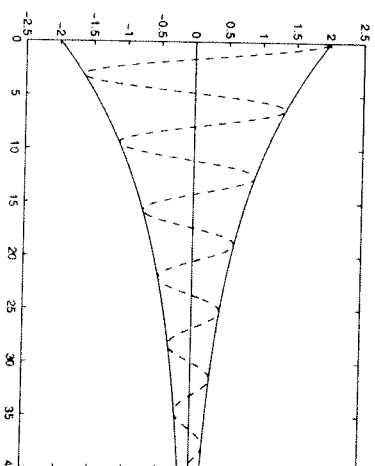
Damped and Free Vibration Example



$$u'' + 0.125u' + u = 0, u'' + u = 0, u(0) = 2, u'(0) = 0$$

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Undamped and Free Vibration Example



$$u'' + 0.125u' + u = 0, u(0) = 2, u'(0) = 0, \text{ and envelope}$$

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Critical and Overdamped Free Vibration

- Damped oscillation

$$0 < \frac{\gamma^2}{4km} < 1$$
- Critical damping yields no oscillation only asymptotic convergence to equilibrium

$$\frac{\gamma^2}{4km} = 1 \rightarrow \gamma = \sqrt{4km}, \mu = 0$$
- Overdamping when

$$\frac{\gamma^2}{4km} > 1 \quad \text{two distinct negative real roots}$$

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Critically Damped Free Vibration

Textbook page 199

For undamped $u'' + u = 0, k = m = 1 \rightarrow \gamma = 2$ is critical damping

$$u'' + 2u' + u = 0$$

$$-\frac{\gamma}{2m} = -1$$

$$u(t) = e^{-t}(A + Bt)$$

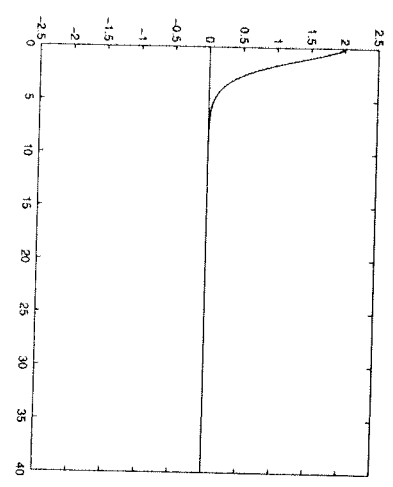
$$u(0) = 2, \quad u'(0) = 0 \rightarrow A = B = 2$$

$$u(t) = 2e^{-t}(1 + t)$$

Comparatively very rapid damping toward equilibrium.

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Critically Damped Free Vibration



$$u'' + 2u' + u = 0, u(0) = 2, u'(0) = 0, \text{ and } u = 2e^{-t}(1+t)$$