Set 14: Vibrations Part 2

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Ordinary Differential Equations
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Mechanical Vibration

Undamped ($\gamma = 0$) and forced ($F(t) \neq 0$) vibrations satisfy

$$mu'' + ku = F(t)$$

- homgeneous solution known $u_h(t) = A \cos \omega_0 t + B \sin \omega_0 t$ with natural frequency $\omega_0^2 = k/m$ and period $T = 2\pi/\omega_0$
- need a particular solution
- interested in interaction of sinusoidal forcing function and the ODE
- $F(t) = F_0 \cos \omega t$, relate behavior to ω and ω_0.

Particular Solution

$$mu'' + ku = F_0 \cos \omega t, \quad \omega \neq \omega_0$$

$$U(t) = A \cos \omega t, \quad U' = -A\omega \sin \omega t, \quad U'' = -A\omega^2 \cos \omega t$$

$$-mA\omega^2 \cos \omega t - kA \cos \omega t = F_0 \cos \omega t$$

$$(k - m\omega^2)A = F_0 \to A = \frac{F_0}{(k - m\omega^2)} = \frac{F_0}{m(\omega_0^2 - \omega^2)}$$

$$u(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t$$

General Solution

$$u(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t$$

$$u'(t) = -c_1 \omega_0 \sin \omega_0 t + c_2 \omega_0 \cos \omega_0 t - \frac{F_0 \omega}{m(\omega_0^2 - \omega^2)} \sin \omega t$$

- sum of two oscillations
- different frequencies
- different amplitudes

Response to $F(t)$

- Start mass/spring system “at rest”
- input $F(t) = \cos \omega t$ with $\omega \neq \omega_0$
- all energy in the response comes from $F(t)$
- $u(t)$ is the response to $F(t)$ with $u(0) = u'(0) = 0$
- sum of oscillations with same amplitude but two frequencies

$$u(0) = c_1 + \frac{F_0}{m(\omega_0^2 - \omega^2)} = 0, \qquad u'(0) = c_2\omega_0 = 0$$

$$u(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)} \left[\cos \omega t - \cos \omega_0 t\right]$$

Some Trigonometry

$$\omega_0 t = \frac{(\omega_0 + \omega)t}{2} + \frac{(\omega_0 - \omega)t}{2} = X + Y$$

$$\omega t = \frac{(\omega_0 + \omega)t}{2} - \frac{(\omega_0 - \omega)t}{2} = X - Y$$

$$\cos(X - Y) = \left[\cos X \cos Y + \sin X \sin Y\right]$$

$$\cos(X + Y) = \left[\cos X \cos Y - \sin X \sin Y\right]$$

$$\cos(X - Y) - \cos(X + Y) = 2 \sin X \sin Y$$

Response to $F(t)$

$$u(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)} \left[\cos \omega t - \cos \omega_0 t \right]$$

$$u(t) = \Big[\frac{2F_0}{m(\omega_0^2 - \omega^2)} \sin\Big(\frac{(\omega_0 - \omega)t}{2} \Big) \Big] \sin\Big(\frac{(\omega_0 + \omega)t}{2} \Big)$$

$$u(t) = R(t) \sin \omega_f t$$

An oscillation with its amplitude given by another oscillation.

Response to $F(t)$

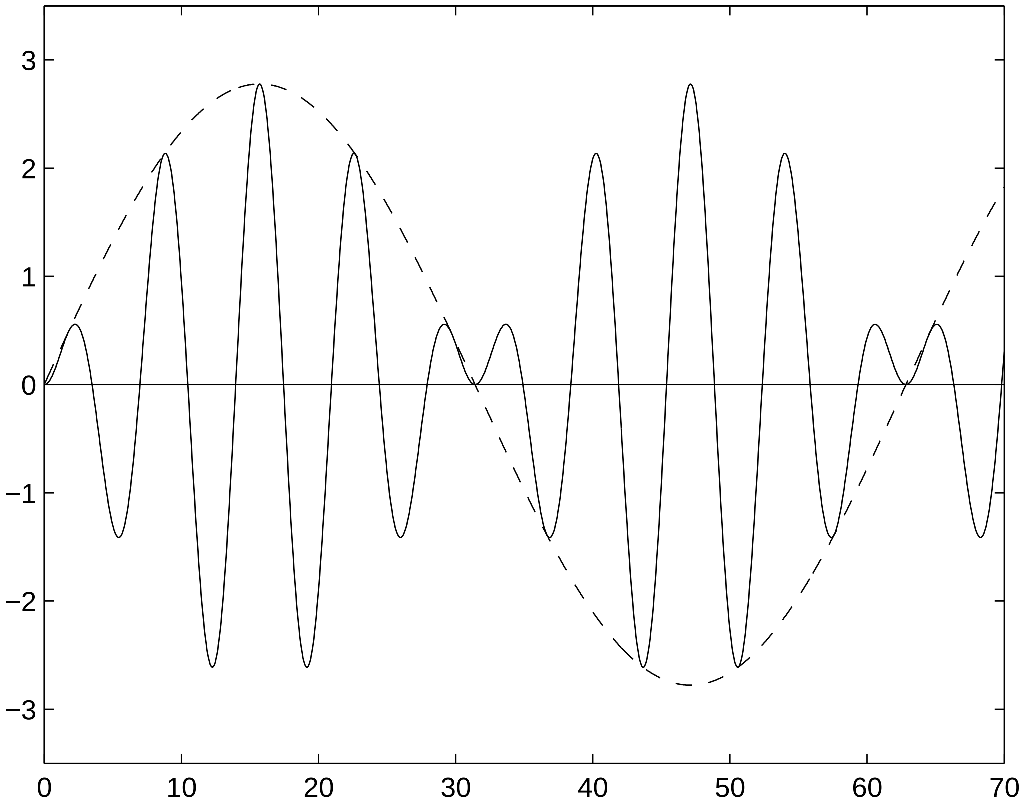
$$|\omega_0 - \omega| << \omega_0 + \omega$$

$$u(t) = [\frac{2F_0}{m(\omega_0^2 - \omega^2)} \sin(\frac{(\omega_0 - \omega)t}{2})] \sin(\frac{(\omega_0 + \omega)t}{2})$$

$$u(t) = R(t) \sin \omega_f t$$

- $\omega_f >> |\omega_0 - \omega|$
- “fast” oscillation $\sin \omega_f t$ with amplitude varying via “slow” oscillation $R(t)$.
- amplitude modulation
- beat phenomenon

Beat Phenomenon



$$u'' + u = 0.5\cos 0.8t, \quad u(0) = u'(0) = 0;$$

$u(t) = 2.7778\sin(0.1t)\sin(0.9t)$; page 214

Resonance

Consider “at rest” response and let $\omega \to \omega_0$.

$$\frac{(\cos \omega t - \cos \omega_0 t)}{(\omega_0^2 - \omega^2)} \to \frac{0}{0}$$

differentiate w/r to ω

$$\frac{-t \sin \omega t}{-2\omega} \to \frac{-t \sin \omega_0 t}{-2\omega_0}$$

$$\therefore \lim_{t \to \infty} u(t : \omega_0) \to \infty$$

Unbounded growth when $\omega = \omega_0$? We can solve the ODE to verify.