Set 15: Vibrations Part 3

Kyle A. Gallivan

Department of Mathematics

Florida State University

Ordinary Differential Equations
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Resonance

Consider "at rest" response and let $\omega \to \omega_0$.

$$mu'' + ku = F_0 \cos \omega t$$

$$\frac{(\cos \omega t - \cos \omega_0 t)}{(\omega_0^2 - \omega^2)} \to \frac{0}{0}$$

differentiate w/r to ω

$$\frac{-t \sin \omega t}{-2\omega} \to \frac{-t \sin \omega_0 t}{-2\omega_0}$$

$$\therefore \lim_{t \to \infty} u(t : \omega_0) \to \infty$$

We can solve the ODE to verify for any initial conditions.

Particular Solution

$$mu'' + ku = F_0 \cos \omega_0 t$$

$$u_h(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t$$

$$U(t) = A \cos \omega_0 t, \quad \text{is a homogeneous solution}$$

$$\therefore U(t) = At \cos \omega_0 t + Bt \sin \omega_0 t$$

Particular Solution

Using Undetermined Coefficients yields the equation in which we equate coefficients and solve:

$$F_0 \cos \omega_0 t = \cos \omega_0 t \left[2m\omega_0 B \right] + \sin \omega_0 t \left[-2m\omega_0 A \right]$$

$$+ t \cos \omega_0 t \left[Ak - Am\omega_0^2 \right] + t \sin \omega_0 t \left[Bk - Bm\omega_0^2 \right]$$

$$\Downarrow$$

$$A = 0, \quad B = \frac{F_0}{2m\omega_0}$$

$$U(t) = \frac{F_0}{2m\omega_0} t \sin \omega_0 t$$

General Solution

$$mu'' + ku = F_0 \cos \omega_0 t$$

$$u_h(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t$$

$$U(t) = \frac{F_0}{2m\omega_0} \sin \omega_0 t$$

$$u(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \frac{F_0}{2m\omega_0} t \sin \omega_0 t$$

- homogeneous solution stays bounded for any initial conditions.
- $t \sin \omega_0 t \rightarrow$ an unbounded oscillation.
- no such thing in practice: model or system fails
- need to analyze damping and forcing together

Forced and Damped Vibration

Consider

$$mu'' + \gamma u' + ku = F_0 \cos \omega t$$

$$u_h(t) = c_1 u_1(t) + c_2 u_2(t) \quad \text{homogeneous general solution}$$

$$U(t) = A \cos \omega t + B \sin \omega t \quad \text{particular solution}$$

$$u(t) = u_h(t) + U(t) \quad \text{general solution}$$

- A and B set to make $U(t)$ a particular solution.
- c_1 and c_2 set to satisfy initial conditions.
- u_1 and u_2 depend on roots of characteristic equations.

Homogeneous Solutions

$$m > 0, \quad k > 0, \quad \gamma > 0$$

$$r_{\pm} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}$$

- real part always negative or zero
- $\gamma^2 < 4km$ decaying oscillation
- $\gamma^2 > 4km$ distinct real negative roots since $\gamma > |\sqrt{\gamma^2 - 4mk}|$
- $u_h(t)$ always decays to 0

Forced and Damped Vibration

Consider

$$mu'' + \gamma u' + ku = F_0 \cos \omega t$$

$$u(t) = u_h(t) + U(t) \quad \text{general solution}$$

- u_h is called the transient solution since its influence on the solution decays to irrelevance due to the damping in the system.
- $U(t)$ is called the steady state since eventually $u(t)$ is arbitrarily close to $U(t)$ for any initial conditions.
- $U(t)$ is also called the forced response since the system is “forced” using $F(t)$ to respond eventually with essentially $U(t)$.
- correspondence between $F(t)$ and $U(t)$ characterizes the system.

Forced Response

Textbook page 208

$$U(t) = A\cos\omega t + B\sin\omega t$$
$$= R\cos\omega t - \delta$$

$$\omega_0^2 = \frac{k}{m}, \quad \Delta^2 = m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2$$

$$\cos\delta = \frac{m(\omega_0^2 - \omega^2)}{\Delta}, \quad \sin\delta = \frac{\gamma\omega}{\Delta}$$

$$R = \frac{F_0}{\Delta}$$

Static Displacement

- If a force F_0 is applied to the spring the resulting displacement is F_0/k.
- Called static displacement.
- $F(t)$ has maximum magnitude of F_0 by assumption.
- ω determines how often maximum magnitude force is applied.
- $\omega = 0 \to F(t) = F_0$

$$mu'' + \gamma u' + ku = F(t), \quad u(0) = \frac{F_0}{k}, \quad u'(0) = 0$$

$$u(t) = \frac{F_0}{k} \quad \text{constant response at static displacement}$$

Amplitude of Forced Response

- Find R relative to maximum force F_0 and static displacement.
- Damping, γ, and forcing frequency, ω, determine relationship for fixed k and m.
- $\Gamma = \gamma^2/(km)$: measure of damping in system.
- ω^2/ω_0^2: measure of frequency of forcing relative to natural frequency of system

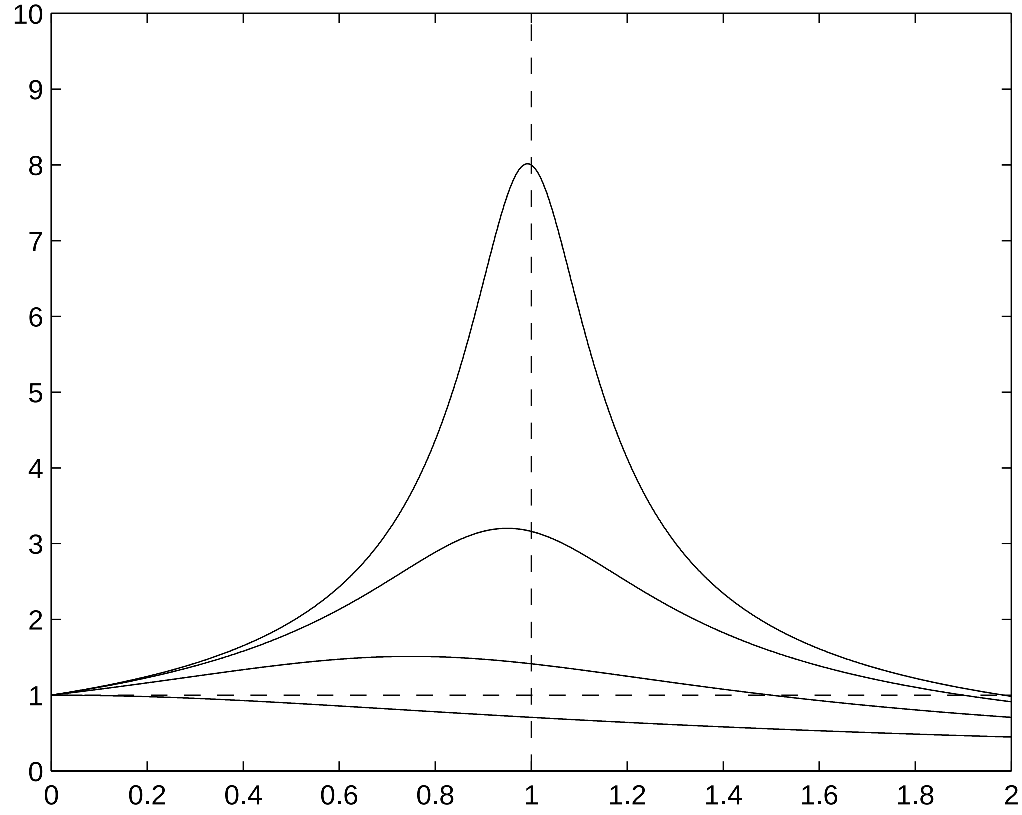
Amplitude of Forced Response

$$\frac{Rk}{F_0} = \left[(1 - \frac{\omega^2}{\omega_0^2})^2 + \Gamma\frac{\omega^2}{\omega_0^2}\right]^{-1/2}$$

Predictions wanted for 3 cases of interest for various levels of damping:

- $\omega^2/\omega_0^2 \rightarrow 0$: low frequency forcing of system
- $\omega^2/\omega_0^2 \rightarrow \infty$: high frequency forcing of system
- $\omega^2/\omega_0^2 \approx 1$: forcing frequency near natural frequency of system

Forced Response Magnitude



Rk/F_0 vs. ω/ω_0 for $\Gamma = 2, 0.5, 0.1, 1/64$

Forced Response Magnitude

- $\omega^2/\omega_0^2 \rightarrow 0$: $R \rightarrow F_0/k$ static displacement
- $\omega^2/\omega_0^2 \rightarrow \infty$: $R \rightarrow 0$
- for small γ and $\omega^2/\omega_0^2 \rightarrow 1$: $R \approx F_0/(\gamma\omega_0)$
- For light damping the maximum magnitude can be very large even for small magnitude forcing.
- For fixed damping $\omega_{max} < \omega_0$ where R_{max} occurs at ω_{max}.
- $\omega_{max} \rightarrow \omega_0$ and $R_{max} \rightarrow \infty$ as $\gamma \rightarrow 0$.
- If $\Gamma = 0$ vertical asymptote at $\omega^2/\omega_0^2 = 1$.

Forced Response Phase

$$\Delta^2 = m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2$$

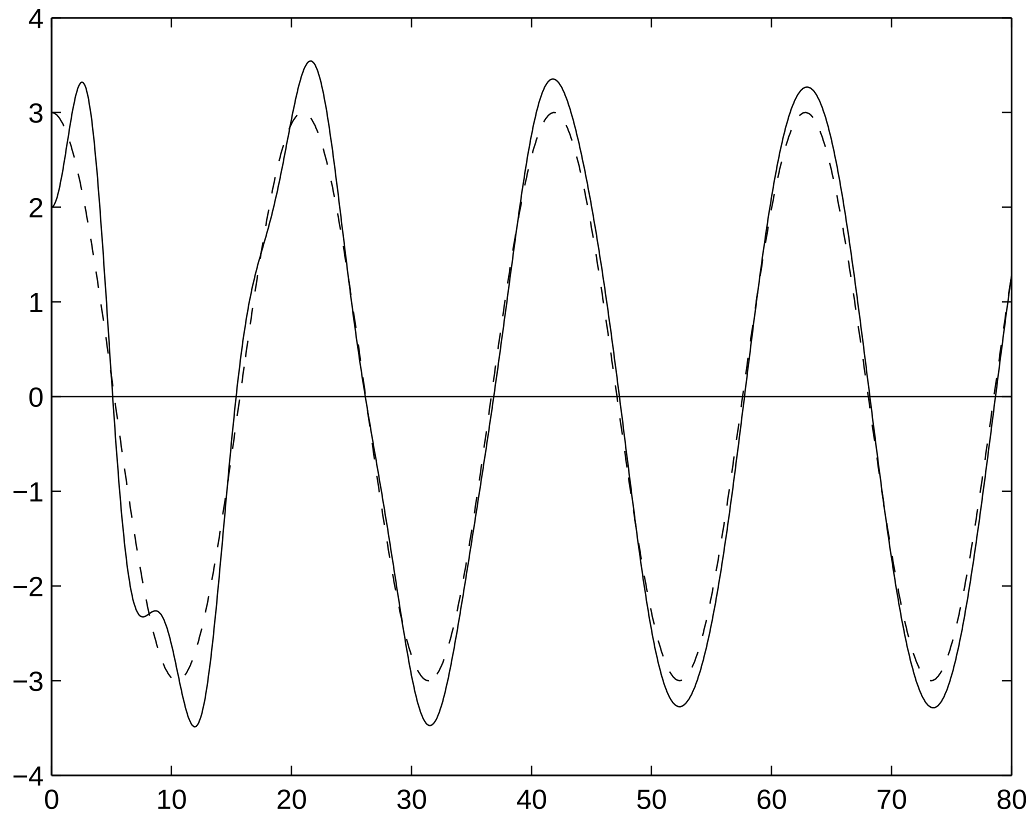
$$\cos\delta = \frac{m(\omega_0^2 - \omega^2)}{\Delta}, \quad \sin\delta = \frac{\gamma\omega}{\Delta}$$

- $\omega \approx 0 \to \cos\delta \approx 1, \quad \sin\delta \approx 0, \quad \delta \approx 0$: forcing function and response very close in phase, i.e., peaks and valleys close in time.
- $\omega = \omega_0 \to \delta = \pi/2$: peaks and valleys of response lags behind those of the forcing function by $\pi/2$.

Forced Response Phase

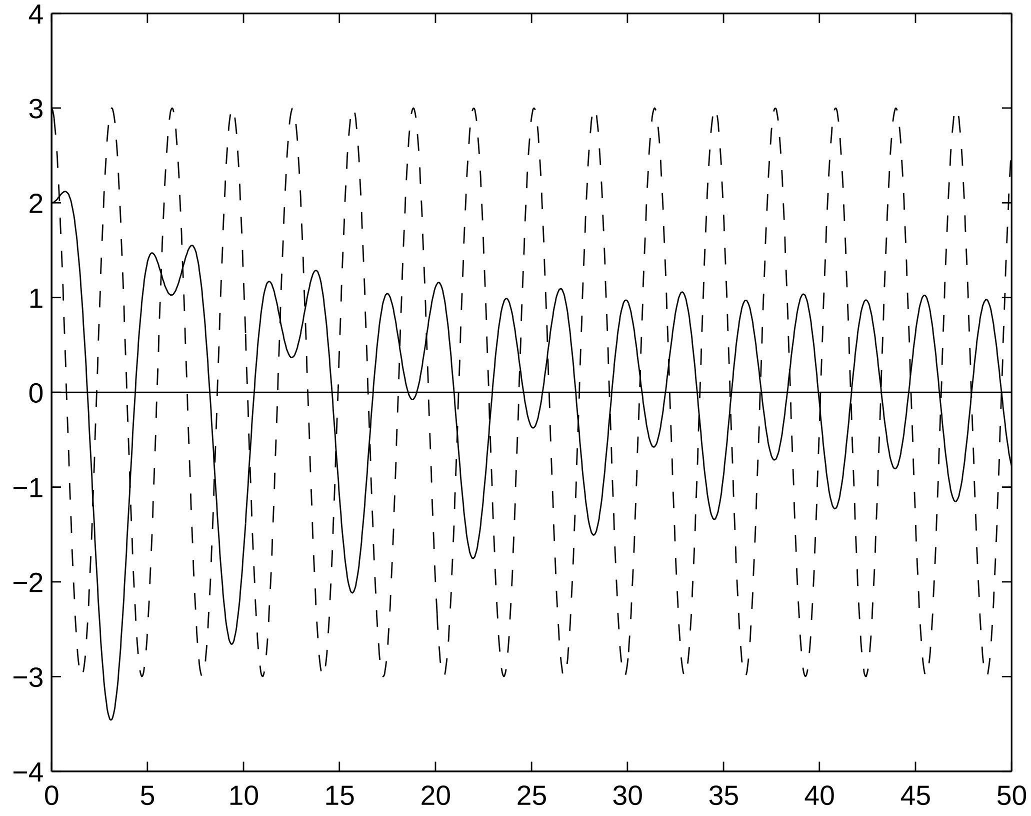
- $\omega >> \omega_0 \rightarrow \delta \approx \pi$ and the response is almost completely out of phase with the forcing function, i.e., peaks of one occur at nearly the same time as valleys of the other.
- for light damping the change from in phase to out of phase happens abruptly
- for heavy damping the change from in phase to out of phase happens slowly

Lightly Damped Low Frequency Forcing Response



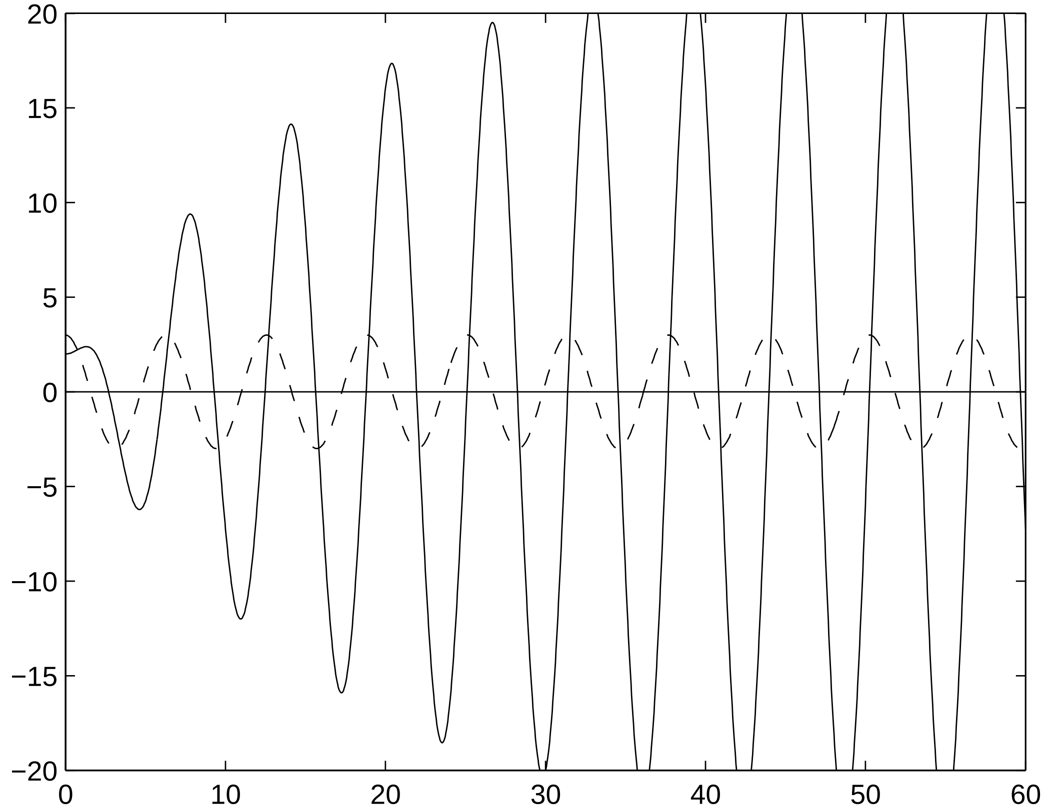
$$u'' + 0.125u' + u = 3\cos 0.3t,\ u_0 = 2,\ v_0 = 0$$

Lightly Damped High Frequency Forcing Response



$$u'' + 0.125u' + u = 3\cos 2t,\; u_0 = 2,\; v_0 = 0$$

Lightly Damped Resonant Forcing Response



$$u'' + 0.125u' + u = 3\cos t,\ u_0 = 2,\ v_0 = 0$$

Summary for Free Vibrations

- homogeneous second order linear ODE
- unforced free vibrations yield a steady oscillation at the natural frequency $\omega_0 = \sqrt{k/m}$
- amplitude-phase form uses maximum amplitude R and phase angle δ given by initial conditions:

$$R \cos(\omega_0 t - \delta)$$

- damping $\gamma > 0$ yields
  - decaying ocsillation when complex conjugate roots $\gamma^2 < 4km$;
  - non-oscillation decaying exponentially to 0 when critically damped (repeated negative real root) $\gamma^2 = 4km$;
  - non-oscillation decaying exponentially to 0 when overdamped (distinct negative real roots). $\gamma^2 > 4km$;

Summary for Forced Vibrations

- Forced undamped vibration with $\omega \neq \omega_0$ yields a beat
- Forced undamped vibration with $\omega = \omega_0$ yields unbounded resonance
- Forced damped vibration yields transient response from homogeneous solution and steady state or forced response from particular solution
- Relationship between amplitude and phase of $F(t)$ and the amplitude and phase of $U(t)$ depends on amount of damping and type of excitation.