

## Set 15: Vibrations Part 3

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Ordinary Differential Equations

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### Particular Solution

$$mu'' + ku = F_0 \cos \omega_0 t$$

$$u_h(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t$$

$U(t) = A \cos \omega_0 t$ , is a homogeneous solution

$$\therefore U(t) = At \cos \omega_0 t + Bt \sin \omega_0 t$$

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### Resonance

Consider "at rest" response and let  $\omega \rightarrow \omega_0$ .

$$\begin{aligned} mu'' + ku &= F_0 \cos \omega t \\ \frac{(\cos \omega t - \cos \omega_0 t)}{(\omega_0^2 - \omega^2)} &\rightarrow \frac{0}{0} \end{aligned}$$

$$\begin{aligned} \text{differentiate w/r to } \omega \\ \frac{-t \sin \omega t}{-2\omega} \rightarrow \frac{-t \sin \omega_0 t}{-2\omega_0} \end{aligned}$$

$$\therefore \lim_{t \rightarrow \infty} u(t; \omega_0) \rightarrow \infty$$

We can solve the ODE to verify for any initial conditions.

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### Particular Solution

Using Undetermined Coefficients yields the equation in which we equate coefficients and solve:

$$\begin{aligned} F_0 \cos \omega_0 t &= \cos \omega_0 t [2m\omega_0 B] + \sin \omega_0 t [-2m\omega_0 A] \\ &+ t \cos \omega_0 t [Ak - Am\omega_0^2] + t \sin \omega_0 t [Bk - Bm\omega_0^2] \end{aligned}$$

$\Downarrow$

$$A = 0, \quad B = \frac{F_0}{2m\omega_0}$$

$$U(t) = \frac{F_0}{2m\omega_0} t \sin \omega_0 t$$

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### General Solution

$$mu'' + ku = F_0 \cos \omega_0 t$$

$$u_h(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t$$

$$U(t) = \frac{F_0}{2m\omega_0} \sin \omega_0 t$$

$$u(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \frac{F_0}{2m\omega_0} t \sin \omega_0 t$$

- homogeneous solution stays bounded for any initial conditions.
- $t \sin \omega_0 t \rightarrow$  an unbounded oscillation.
- no such thing in practice: model or system fails
- need to analyze damping and forcing together

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### Forced and Damped Vibration

Consider

$$mu'' + \gamma u' + ku = F_0 \cos \omega t$$

$$u_h(t) = c_1 u_1(t) + c_2 u_2(t) \quad \text{homogeneous general solution}$$

$$U(t) = A \cos \omega t + B \sin \omega t \quad \text{particular solution}$$

$$u(t) = u_h(t) + U(t) \quad \text{general solution}$$

- $A$  and  $B$  set to make  $U(t)$  a particular solution.
- $c_1$  and  $c_2$  set to satisfy initial conditions.
- $u_1$  and  $u_2$  depend on roots of characteristic equations.

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### Homogeneous Solutions

$$m > 0, \quad k > 0, \quad \gamma > 0$$

$$r_{\pm} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}$$

- real part always negative or zero
- $\gamma^2 < 4km$  decaying oscillation
- $\gamma^2 > 4km$  distinct real negative roots since  $\gamma > |\sqrt{\gamma^2 - 4mk}|$
- $u_h(t)$  always decays to 0

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### Forced and Damped Vibration

Consider

$$mu'' + \gamma u' + ku = F_0 \cos \omega t$$

$$u(t) = u_h(t) + U(t) \quad \text{general solution}$$

- $u_h$  is called the transient solution since its influence on the solution decays to irrelevance due to the damping in the system.
- $U(t)$  is called the steady state since eventually  $u(t)$  is arbitrarily close to  $U(t)$  for any initial conditions.
- $U(t)$  is also called the forced response since the system is "forced" using  $F(t)$  to respond eventually with essentially  $U(t)$ .
- correspondence between  $F(t)$  and  $U(t)$  characterizes the system.

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**Forced Response**

$$U(t) = A \cos \omega t + B \sin \omega t \\ = R \cos \omega t - \delta$$

$$\omega_0^2 = \frac{k}{m}, \quad \Delta^2 = m^2(\omega_0^2 - \omega^2)^2 + \gamma^2 \omega^2$$

$$\cos \delta = \frac{m(\omega_0^2 - \omega^2)}{\Delta}, \quad \sin \delta = \frac{\gamma \omega}{\Delta}$$

$$R = \frac{F_0}{\Delta}$$

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**Amplitude of Forced Response**

- Find  $R$  relative to maximum force  $F_0$  and static displacement.
- Damping,  $\gamma$ , and forcing frequency,  $\omega$ , determine relationship for fixed  $k$  and  $m$ .
- $\Gamma = \gamma^2/(km)$ : measure of damping in system.
- $\omega^2/\omega_0^2$ : measure of frequency of forcing relative to natural frequency of system

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**Static Displacement**

- If a force  $F_0$  is applied to the spring the resulting displacement is  $F_0/k$ .
- Called static displacement.
- $F(t)$  has maximum magnitude of  $F_0$  by assumption.
- $\omega$  determines how often maximum magnitude force is applied.
- $\omega = 0 \rightarrow F(t) = F_0$

$$m u'' + \gamma u' + k u = F(t), \quad u(0) = \frac{F_0}{k}, \quad u'(0) = 0$$

$$u(t) = \frac{F_0}{k} \quad \text{constant response at static displacement}$$

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**Amplitude of Forced Response**

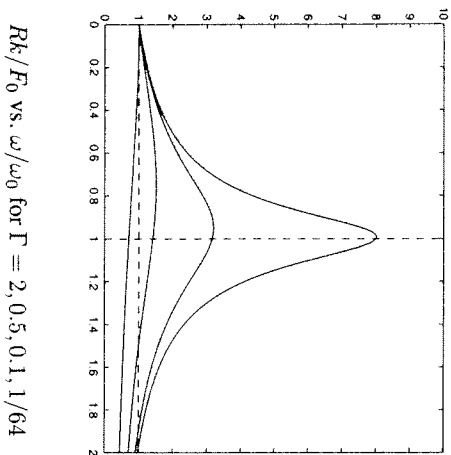
Predictions wanted for 3 cases of interest for various levels of damping:

$$\frac{Rk}{F_0} = \left[ \left( 1 - \frac{\omega^2}{\omega_0^2} \right)^2 + \Gamma \frac{\omega^2}{\omega_0^2} \right]^{-1/2}$$

- $\omega^2/\omega_0^2 \rightarrow 0$ : low frequency forcing of system
- $\omega^2/\omega_0^2 \rightarrow \infty$ : high frequency forcing of system
- $\omega^2/\omega_0^2 \approx 1$ : forcing frequency near natural frequency of system

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### Forced Response Magnitude



$Rk/F_0$  vs.  $\omega/\omega_0$  for  $\Gamma = 2, 0.5, 0.1, 1/64$

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### Forced Response Magnitude

- $\omega^2/\omega_0^2 \rightarrow 0: R \rightarrow F_0/k$  static displacement
- $\omega^2/\omega_0^2 \rightarrow \infty: R \rightarrow 0$
- for small  $\gamma$  and  $\omega^2/\omega_0^2 \rightarrow 1: R \approx F_0/(\gamma\omega_0)$
- For light damping the maximum magnitude can be very large even for small magnitude forcing.
- For fixed damping  $\omega_{max} < \omega_0$  where  $R_{max}$  occurs at  $\omega_{max}$ .
- $\omega_{max} \rightarrow \omega_0$  and  $R_{max} \rightarrow \infty$  as  $\gamma \rightarrow 0$ .
- If  $\Gamma = 0$  vertical asymptote at  $\omega^2/\omega_0^2 = 1$ .

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### Forced Response Phase

$$\Delta^2 = m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2$$

$$\cos \delta = \frac{m(\omega_0^2 - \omega^2)}{\Delta}, \quad \sin \delta = \frac{\gamma\omega}{\Delta}$$

- $\omega \approx 0 \rightarrow \cos \delta \approx 1, \quad \sin \delta \approx 0, \quad \delta \approx 0$ : forcing function and response very close in phase, i.e., peaks and valleys close in time.
- $\omega = \omega_0 \rightarrow \delta = \pi/2$ : peaks and valleys of response lags behind those of the forcing function by  $\pi/2$ .

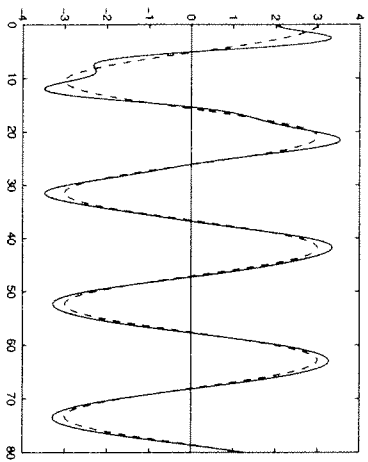
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### Forced Response Phase

- $\omega \gg \omega_0 \rightarrow \delta \approx \pi$  and the response is almost completely out of phase with the forcing function, i.e., peaks of one occur at nearly the same time as valleys of the other.
- for light damping the change from in phase to out of phase happens abruptly
- for heavy damping the change from in phase to out of phase happens slowly

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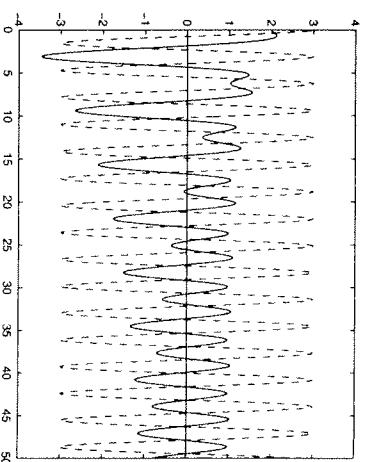
### Lightly Damped Low Frequency Forcing Response



$$u'' + 0.125u' + u = 3 \cos 0.3t, u_0 = 2, v_0 = 0$$

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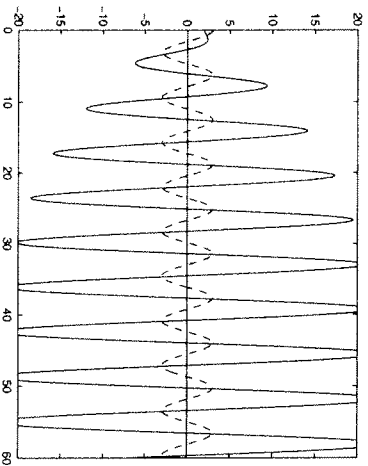
### Lightly Damped High Frequency Forcing Response



$$u'' + 0.125u' + u = 3 \cos 2t, u_0 = 2, v_0 = 0$$

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### Lightly Damped Resonant Forcing Response



$$u'' + 0.125u' + u = 3 \cos t, u_0 = 2, v_0 = 0$$

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### Summary for Free Vibrations

- homogeneous second order linear ODE
- unforced free vibrations yield a steady oscillation at the natural frequency  $\omega_0 = \sqrt{k/m}$
- amplitude-phase form uses maximum amplitude  $R$  and phase angle  $\delta$  given by initial conditions:

$$R \cos(\omega_0 t - \delta)$$

- damping  $\gamma > 0$  yields
  - decaying oscillation when complex conjugate roots  $\gamma^2 < 4km$ ;
  - non-oscillation decaying exponentially to 0 when critically damped (repeated negative real root)  $\gamma^2 = 4km$ ;
  - non-oscillation decaying exponentially to 0 when overdamped (distinct negative real roots);  $\gamma^2 > 4km$ ;

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### Summary for Forced Vibrations

- Forced undamped vibration with  $\omega \neq \omega_0$  yields a beat
- Forced undamped vibration with  $\omega = \omega_0$  yields unbounded resonance
- Forced damped vibration yields transient response from homogeneous solution and steady state or forced response from particular solution
- Relationship between amplitude and phase of  $F(t)$  and the amplitude and phase of  $U(t)$  depends on amount of damping and type of excitation: