

Set 16: Laplace Transform and IVPs Part 1

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Ordinary Differential Equations

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Integral Transforms

Consider a mapping of a function $f(t)$ to another function $F(s)$:

$$F(s) = \int_{\alpha}^{\beta} K(s, t) f(t) dt$$

- $K(s, t)$ is the kernel of the transform.
- α, β may be finite or infinite
- improper integrals may be needed
- has the flavor of an infinite dimensional matrix times a vector

Integral Transforms and IVPs

- We want to transform the IVP from a differential equation into an algebraic equation.
- Form of ODE determines the choice of integral transform.
- Effect of integral transform on derivatives must be considered.
- Kernel relates to fundamental solutions.
- We will consider second order constant coefficient linear ODEs.

Laplace Transform

Consider a mapping of a function $f(t)$ to another function $F(s)$:

$$F(s) = \int_0^{\infty} e^{-st} f(t) dt$$

- $K(s, t) = e^{-st}$
- $\alpha = 0, \beta = \infty$
- improper integral
- a special case of a definition over \mathbb{C}

Improper Integral

$$\begin{aligned}\int_a^{\infty} g(t) dt &= \lim_{A \rightarrow \infty} \int_a^A g(t) dt \\ &= \lim_{A \rightarrow \infty} [G(A) - G(a)]\end{aligned}$$

- $G(t)$ is an antiderivative of $g(t)$
- integral must exist for each $A > a$
- limit as $A \rightarrow \infty$ must exist
- if both exist then the improper integral converges
- if not then the improper integral diverges

Improper Integral

$$\begin{aligned}\int_a^\infty \frac{1}{t} dt &= \lim_{A \rightarrow \infty} \int_a^A \frac{1}{t} dt \\ &= \lim_{A \rightarrow \infty} [\ln A - \ln a] \\ &= \infty\end{aligned}$$

- $G(t)$ exists
- $\lim_{A \rightarrow \infty} [G(A) - G(a)]$ does not
- improper integral diverges

Improper Integral

$$\int_1^{\infty} t^{-p} dt, \quad p \neq 1$$
$$\int_1^{\infty} t^{-p} dt = \lim_{A \rightarrow \infty} \int_1^A t^{-p} dt$$
$$= \lim_{A \rightarrow \infty} \frac{1}{1-p} [t^{1-p}]_1^A$$
$$= \lim_{A \rightarrow \infty} \frac{1}{1-p} [A^{1-p} - 1]$$

- if $p > 1 \rightarrow A^{1-p} \rightarrow 0$ and $\int_1^{\infty} t^{-p} dt = 1/(p-1)$
- $p < 1 \rightarrow A^{1-p} \rightarrow \infty$
- if $p < 1$ the improper integral diverges
- $p = 1$ already seen to diverge.

Laplace Transform

- various sufficient conditions for convergence of improper integrals (e.g., Theorem 6.1.1. page 307)
- Laplace works for piecewise continuous and continuous functions of exponential order

Theorem 16.1 (Textbook page 308). *The Laplace transform $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > a$ if*

- *f is piecewise continuous on $0 \leq t \leq A$ for any positive A*
- *$\exists K > 0$, $\exists M > 0$ and $\exists a \in \mathbb{R}$ such that $|f(t)| \leq Ke^{at}$, $t \geq M$.*

Example

$$\begin{aligned}\mathcal{L}\{1\} &= \int_0^{\infty} e^{-st} dt \\ &= \lim_{A \rightarrow \infty} -\frac{1}{s} [e^{-st}]_0^A \\ &= \lim_{A \rightarrow \infty} -\frac{1}{s} [e^{-sA} - 1] \\ &= \frac{1}{s}, \quad s > 0\end{aligned}$$

Example

$$\begin{aligned}\mathcal{L}\{e^{at}\} &= \int_0^{\infty} e^{-(s-a)t} dt \\ &= \lim_{A \rightarrow \infty} -\frac{1}{s-a} [e^{-(s-a)t}]_0^A \\ &= \lim_{A \rightarrow \infty} -\frac{1}{s-a} [e^{-(s-a)A} - 1] \\ &= \frac{1}{s-a}, \quad s > a\end{aligned}$$

Example

$$\mathcal{L}\{\sin at\} = \int_0^{\infty} e^{-st} \sin at \, dt, \quad s > 0$$

$$F(s) = \lim_{A \rightarrow \infty} \int_0^A e^{-st} \sin at \, dt$$

$$u = e^{-st}, \quad u' = -se^{-st}, \quad v' = \sin at, \quad v = -(\cos at)/a$$

$$\begin{aligned} F(s) &= \lim_{A \rightarrow \infty} \left[-\frac{e^{-st} \cos at}{a} \Big|_0^A - \frac{s}{a} \int_0^A e^{-st} \cos at \, dt \right] \\ &= \frac{1}{a} - \frac{s}{a} \int_0^{\infty} e^{-st} \cos at \, dt \end{aligned}$$

Example

$$F(s) = \frac{1}{a} - \frac{s}{a} \int_0^{\infty} e^{-st} \cos at \, dt$$

$$= \frac{1}{a} - \frac{s}{a} \lim_{A \rightarrow \infty} \int_0^A e^{-st} \cos at \, dt$$

$$u = e^{-st}, \quad u' = -se^{-st}, \quad v' = \cos at, \quad v = (\sin at)/a$$

$$= \frac{1}{a} - \frac{s}{a} \lim_{A \rightarrow \infty} \left\{ \left[\frac{e^{-st} \sin at}{a} \right]_0^A + \frac{s}{a} \int_0^A e^{-st} \sin at \, dt \right\}$$

$$= \frac{1}{a} - \frac{s}{a} \left[\frac{s}{a} F(s) \right]$$

$$F(s) = \frac{a}{s^2 + a^2}, \quad s > 0$$

Piecewise Continuous Example

$$f(t) = \begin{cases} 1 & 0 \leq t < 1 \\ k & t = 1 \\ 0 & t > 1 \end{cases}$$

$$\begin{aligned} \mathcal{L}\{f\} = F(s) &= \int_0^{\infty} e^{-st} f(t) dt \\ &= \int_0^1 e^{-st} dt + \int_1^{\infty} e^{-st} \times 0 dt \\ &= \int_0^1 e^{-st} dt = -\left[\frac{e^{-st}}{s}\right]_0^1 = \frac{1 - e^{-s}}{s}, \quad s > 0 \end{aligned}$$

- $F(s)$ is independent of k .
- If $f - g$ is piecewise continuous then $\mathcal{L}\{f\} = \mathcal{L}\{g\}$

Two Important Properties

- The Laplace transform is linear:

$$\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\} = c_1 \mathcal{L}\{f_1(t)\} + c_2 \mathcal{L}\{f_2(t)\}$$

- Follows from linearity of integration.
- The Laplace transform of a function and its derivative are related simply.

Two Important Properties

Theorem 16.2 (Textbook page 313). *Let f be a continuous function on $0 \leq t \leq A$ and let f' be piecewise continuous on the same interval. If $\exists K > 0$, $\exists M > 0$ and $\exists a \in \mathbb{R}$ such that $|f(t)| \leq Ke^{at}$, $t \geq M$ then*

- $\mathcal{L}\{f\}$ and $\mathcal{L}\{f'\}$ exist for $s > a$.
- The two transforms are related by

$$\mathcal{L}\{f'\} = s\mathcal{L}\{f\} - f(0)$$

Two Important Properties

Corollary 16.3. *Let f and f' be continuous functions on $0 \leq t \leq A$ and let f'' be piecewise continuous on the same interval. If $\exists K > 0$, $\exists M > 0$ and $\exists a \in \mathbb{R}$ such that $|f(t)| \leq Ke^{at}$, $t \geq M$ then*

- $\mathcal{L}\{f'\}$ and $\mathcal{L}\{f''\}$ exist for $s > a$.
- The two transforms are related by

$$\mathcal{L}\{f''\} = s^2 \mathcal{L}\{f\} - sf(0) - f'(0)$$

- This generalizes to the n -th derivative. (see Textbook page 314)

Example IVP

$$y'' - y' - 2y = 0, \quad y(0) = 1, \quad y'(0) = 0$$

Denote $\mathcal{L}\{y\}$ as $Y(s)$

$$\mathcal{L}\{y'' - y' - 2y\} = \mathcal{L}\{0\}$$

$$\mathcal{L}\{y''\} - \mathcal{L}\{y'\} - \mathcal{L}\{2y\} = 0$$

$$[s^2Y(s) - sy(0) - y'(0)] - [sY(s) - y(0)] - 2Y(s) = 0$$

Example IVP

$$[sY(s) - y(0)] - 2Y(s) = 0$$

$$(s^2 - s - 2)Y(s) + (1 - s)y(0) - y'(0) = 0$$

$$\begin{aligned} Y(s) &= \frac{(s - 1)}{(s^2 - s - 2)} = \frac{(s - 1)}{(s - 2)(s + 1)} \\ &= \frac{1}{3} \times \frac{1}{s - 2} + \frac{2}{3} \times \frac{1}{s + 1} \end{aligned}$$

using partial fraction expansion.

Example IVP

$$Y(s) = \frac{1}{3} \times \frac{1}{s-2} + \frac{2}{3} \times \frac{1}{s+1}$$

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}, \quad s > a$$

$$\therefore y(t) = \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t}$$

This solves the IVP.

General Form

$$ay'' + by' + cy = f(t), y(0) = y_0, \quad y'(0) = y'_0$$

\Updownarrow

$$Y(s) = \frac{(as + b)y_0 + ay'_0}{as^2 + bs + c} + \frac{F(s)}{as^2 + bs + c}$$

where $F(s) = \mathcal{L}\{f\}$.

Comments

- A differential equation is replaced by an algebraic equation.
- Initial conditions are handled automatically.
- Homogeneous and nonhomogeneous ODEs are handled in the same manner.
- For continuous solutions, $y(t)$, the Laplace transform, $Y(s)$ is unique.
- Linearity can be used for $F(s) = F_1(s) + F_2(s) + \cdots + F_k(s)$:

$$f(t) = \mathcal{L}^{-1}\{F_1(s)\} + \cdots + \mathcal{L}^{-1}\{F_k(s)\}$$

- inversion has a general formula on \mathbb{C} but for \mathbb{R} we will use tables of functions and their transforms.

Example

$$y'' + y = \sin 2t, \quad y_0 = 2, \quad y'_0 = 1$$

$$a = 1 = c, \quad b = 0$$

$$Y(s) = \frac{2s + 1}{s^2 + 1} + \frac{F(s)}{s^2 + 1} = \frac{2s + 1}{s^2 + 1} + \frac{2}{(s^2 + 4)(s^2 + 1)}$$

$$= \frac{2s^3 + s^2 + 8s + 6}{(s^2 + 4)(s^2 + 1)}$$

Example

$$Y(s) = \frac{2s^3 + s^2 + 8s + 6}{(s^2 + 4)(s^2 + 1)}$$
$$= \frac{2s + 5/3}{s^2 + 1} - \frac{2/3}{s^2 + 4} = \frac{2s}{s^2 + 1} + \frac{5/3}{s^2 + 1} - \frac{2/3}{s^2 + 4}$$

$$\frac{s}{s^2 + \omega^2} \leftrightarrow \cos \omega t, \quad \frac{\omega}{s^2 + \omega^2} \leftrightarrow \sin \omega t$$

$$y(t) = 2 \cos t + \frac{5}{3} \sin t - \frac{1}{3} \sin 2t$$