Set 16: Laplace Transform and IVPs Part 1

Kyle A. Gallivan

Department of Mathematics

Florida State University

Ordinary Differential Equations
Fall 2009

1

Integral Transforms

Consider a mapping of a function $f(t)$ to another function $F(s)$:

$$F(s) = \int_{\alpha}^{\beta} K(s,t)\, f(t)\, dt$$

- $K(s,t)$ is the kernel of the transform.
- α, β may be finite or infinite
- improper integrals may be needed
- has the flavor of an infinite dimensional matrix times a vector

2

Integral Transforms and IVPs

- We want to transform the IVP from a differential equation into an algebraic equation.
- Form of ODE determines the choice of integral transform.
- Effect of integral transform on derivatives must be considered.
- Kernel relates to fundamental solutions.
- We will consider second order constant coefficient linear ODEs.

3

Laplace Transform

Consider a mapping of a function $f(t)$ to another function $F(s)$:

$$F(s) = \int_{0}^{\infty} e^{-st}\, f(t)\, dt$$

- $K(s,t) = e^{-st}$
- $\alpha = 0, \beta = \infty$
- improper integral
- a special case of a definition over $\mathbb{C}$

4

Improper Integral

$$\int_a^\infty g(t)\,dt = \lim_{A\to\infty}\int_a^A g(t)\,dt$$
$$= \lim_{A\to\infty}\left[G(A) - G(a)\right]$$

- $G(t)$ is an antiderivative of $g(t)$
- integral must exist for each $A > a$
- limit as $A \to \infty$ must exist
- if both exist then the improper integral converges
- if not then the improper integral diverges

Improper Integral

$$\int_a^\infty \frac{1}{t}\,dt = \lim_{A\to\infty}\int_a^A \frac{1}{t}\,dt$$
$$= \lim_{A\to\infty}\left[\ln A - \ln a\right]$$
$$= \infty$$

- $G(t)$ exists
- $\lim_{A\to\infty}[G(A) - G(a)]$ does not
- improper integral diverges

Improper Integral

$$\int_1^\infty t^{-p}\,dt, \quad p \neq 1$$

$$\int_1^\infty t^{-p}\,dt = \lim_{A\to\infty}\int_1^A t^{-p}\,dt$$
$$= \lim_{A\to\infty}\frac{1}{1-p}\left[t^{1-p}\right]_1^A$$
$$= \lim_{A\to\infty}\frac{1}{1-p}\left[A^{1-p} - 1\right]$$

- if $p > 1 \to A^{1-p} \to 0$ and $\int_1^\infty t^{-p}\,dt = 1/(p-1)$
- $p < 1 \to A^{1-p} \to \infty$
- if $p < 1$ the improper integral diverges
- $p = 1$ already seen to diverge.

Laplace Transform

- various sufficient conditions for convergence of improper integrals (e.g., Theorem 6.1.1. page 307)
- Laplace works for piecewise continuous and contiuous functions of exponential order

Theorem 16.1 (Textbook page 308). *The Laplace transform* $F(s) = \mathcal{L}\{f(t)\}$ *exists for* $s > a$ *if*

- f *is piecewise continuous on* $0 \le t \le A$ *for any positive* A
- $\exists K > 0, \quad \exists M > 0$ *and* $\exists a \in \mathbb{R}$ *such that* $|f(t)| \le Ke^{at}, \quad t \ge M$.

Example

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st}\, dt$$
$$= \lim_{A\to\infty} -\frac{1}{s}\left[e^{-st}\right]_0^A$$
$$= \lim_{A\to\infty} -\frac{1}{s}\left[e^{-sA} - 1\right]$$
$$= \frac{1}{s}, \quad s > 0$$

Example

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-(s-a)t}\, dt$$
$$= \lim_{A\to\infty} -\frac{1}{s-a}\left[e^{-(s-a)t}\right]_0^A$$
$$= \lim_{A\to\infty} -\frac{1}{s-a}\left[e^{-(s-a)A} - 1\right]$$
$$= \frac{1}{s-a}, \quad s > a$$

Example

$$\mathcal{L}\{\sin at\} = \int_0^\infty e^{-st} \sin at\, dt, \quad s > 0$$
$$F(s) = \lim_{A\to\infty} \int_0^A e^{-st} \sin at\, dt$$
$$u = e^{-st}, \quad u' = -se^{-st}, \quad v' = \sin at, \quad v = -(\cos at)/a$$
$$F(s) = \lim_{A\to\infty} \left[-\frac{e^{-st}\cos at}{a}\Big|_0^A - \frac{s}{a}\int_0^A e^{-st}\cos at\, dt\right]$$
$$= \frac{1}{a} - \frac{s}{a}\int_0^\infty e^{-st}\cos at\, dt$$

Example

$$F(s) = \frac{1}{a} - \frac{s}{a}\int_0^\infty e^{-st}\cos at\, dt$$
$$= \frac{1}{a} - \frac{s}{a}\lim_{A\to\infty}\int_0^A e^{-st}\cos at\, dt$$
$$u = e^{-st}, \quad u' = -se^{-st}, \quad v' = \cos at, \quad v = (\sin at)/a$$
$$= \frac{1}{a} - \frac{s}{a}\lim_{A\to\infty}\left\{\left[\frac{e^{-st}\sin at}{a}\right]_0^A + \frac{s}{a}\int_0^A e^{-st}\sin at dt\right\}$$
$$= \frac{1}{a} - \frac{s}{a}\left[\frac{s}{a}F(s)\right]$$
$$F(s) = \frac{a}{s^2+a^2}, \quad s > 0$$

Piecewise Continuous Example

$$f(t) = \begin{cases} 1 & 0 \le t < 1 \\ k & t = 1 \\ 0 & t > 1 \end{cases}$$

$$\mathcal{L}\{f\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^1 e^{-st}\, dt + \int_1^\infty e^{-st} \times 0\, dt$$

$$= \int_0^1 e^{-st}\, dt = -[\frac{e^{-st}}{s}]_0^1 = \frac{1 - e^{-s}}{s}, \quad s > 0$$

- $F(s)$ is independent of k.
- If $f - g$ is piecewise continuous then $\mathcal{L}\{f\} = \mathcal{L}\{g\}$

Two Important Properties

- The Laplace transform is linear:

$$\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\} = c_1\mathcal{L}\{f_1(t)\} + c_2\mathcal{L}\{f_2(t)\}$$

- Follows from linearity of integration.
- The Laplace transform of a function and its derivative are related simply.

Two Important Properties

Theorem 16.2 (Textbook page 313). *Let f be a continuous function on $0 \le t \le A$ and let f' be piecewise continuous on the same interval. If $\exists K > 0, \quad \exists M > 0$ and $\exists a \in \mathbb{R}$ such that $|f(t)| \le Ke^{at}, \quad t \ge M$ then*

- *$\mathcal{L}\{f\}$ and $\mathcal{L}\{f'\}$ exist for $s > a$.*
- *The two transforms are related by*

$$\mathcal{L}\{f'\} = s\mathcal{L}\{f\} - f(0)$$

Two Important Properties

Corollary 16.3. *Let f and f' be continuous functions on $0 \le t \le A$ and let f'' be piecewise continuous on the same interval. If $\exists K > 0, \quad \exists M > 0$ and $\exists a \in \mathbb{R}$ such that $|f(t)| \le Ke^{at}, \quad t \ge M$ then*

- *$\mathcal{L}\{f'\}$ and $\mathcal{L}\{f''\}$ exist for $s > a$.*
- *The two transforms are related by*

$$\mathcal{L}\{f''\} = s^2\mathcal{L}\{f\} - sf(0) - f'(0)$$

- *This generalizes to the n-th derivative. (see Textbook page 314)*

Example IVP

$$y'' - y' - 2y = 0, \quad y(0) = 1, \quad y'(0) = 0$$

Denote $\mathcal{L}\{y\}$ as $Y(s)$

$$\mathcal{L}\{y'' - y' - 2y\} = \mathcal{L}\{0\}$$
$$\mathcal{L}\{y''\} - \mathcal{L}y'\} - \mathcal{L}2y\} = 0$$
$$\left[s^2Y(s) - sy(0) - y'(0)\right] - \left[sY(s) - y(0)\right] - 2Y(s) = 0$$

Example IVP

$$\left[sY(s) - y(0)\right] - 2Y(s) = 0$$

$$(s^2 - s - 2)Y(s) + (1 - s)y(0) - y'(0) = 0$$

$$Y(s) = \frac{(s-1)}{(s^2 - s - 2)} = \frac{(s-1)}{(s-2)(s+1)}$$
$$= \frac{1}{3} \times \frac{1}{s-2} + \frac{2}{3} \times \frac{1}{s+1}$$

using partial fraction expansion.

Example IVP

$$Y(s) = \frac{1}{3} \times \frac{1}{s-2} + \frac{2}{3} \times \frac{1}{s+1}$$
$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}, \quad s > a$$

$$\therefore y(t) = \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t}$$

This solves the IVP.

General Form

$$ay'' + by' + cy = f(t), y(0) = y_0, \quad y'(0) = y_0'$$
$$\Updownarrow$$
$$Y(s) = \frac{(as+b)y_0 + ay_0'}{as^2 + bs + c} + \frac{F(s)}{as^2 + bs + c}$$

where $F(s) = \mathcal{L}\{f\}$.

Comments

- A differential equation is replaced by an algebraic equation.
- Initial conditions are handled automatically.
- Homogeneous and nonhomogeneous ODEs are handled in the same manner.
- For continuous soltions, $y(t)$, the Laplace transform , $Y(s)$ is unique.
- Linearity can be used for $F(s) = F_1(s) + F_2(s) + \cdots + F_k(s)$:

$$f(t) = \mathcal{L}^{-1}\{F_1(s)\} + \cdots + \mathcal{L}^{-1}\{F_k(s)\}$$

- inversion has a general formula on $\mathbb{C}$ but for $\mathbb{R}$ we will use tables of functions and their transforms.

Example

$$y'' + y = \sin 2t, \quad y_0 = 2, \quad y_0' = 1$$
$$a = 1 = c, \quad b = 0$$

$$Y(s) = \frac{2s+1}{s^2+1} + \frac{F(s)}{s^2+1} = \frac{2s+1}{s^2+1} + \frac{2}{(s^2+4)(s^2+1)}$$

$$= \frac{2s^3+s^2+8s+6}{(s^2+4)(s^2+1)}$$

Example

$$Y(s) = \frac{2s^3+s^2+8s+6}{(s^2+4)(s^2+1)}$$
$$= \frac{2s+5/3}{s^2+1} - \frac{2/3}{s^2+4} = \frac{2s}{s^2+1} + \frac{5/3}{s^2+1} - \frac{2/3}{s^2+4}$$

$$\frac{s}{s^2+\omega^2} \leftrightarrow \cos\omega t, \quad \frac{\omega}{s^2+\omega^2} \leftrightarrow \sin\omega t$$

$$y(t) = 2\cos t + \frac{5}{3}\sin t - \frac{1}{3}\sin 2t$$