Set 17: Laplace Transform and IVPs Part 2

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Some Important Properties

Let $\mathcal{L}\{y(t)\} = Y(s)$ and $\mathcal{L}\{f_i(t)\} = F_i(s)$. (Table 6.2.1 in textbook has summary of many imporant properties)

$$\mathcal{L}\{\alpha_1 f_1(t) + \cdots + \alpha_k f_k(t)\} = \alpha_1 F_1(t) + \cdots + \alpha_k F_k(t)$$

$$\mathcal{L}\{y'\} = sY(s) - y(0) \quad \text{and} \quad \mathcal{L}\{y''\} = s^2 Y(s) - sy(0) - y'(0)$$

$$\mathcal{L}\{-tf(t)\} = F'(s) \quad \text{and} \quad g(t) = \int_0^t f(\tau)\, d\tau \Leftrightarrow G(s) = \frac{F(s)}{s}$$

$$f(t) = f(t+T),\ \ t \geq 0,\ \ T > 0 \Leftrightarrow F(s) = \frac{\int_0^T f(t)\, dt}{1 - e^{-sT}}$$

Some Forcing Functions of Interest

- unit step (Heaviside) function for $c \geq 0$
- Dirac delta function
- pulse function
- square wave
- ramp loading
- saw tooth wave
- rectified sine wave

Overview

We will consider:

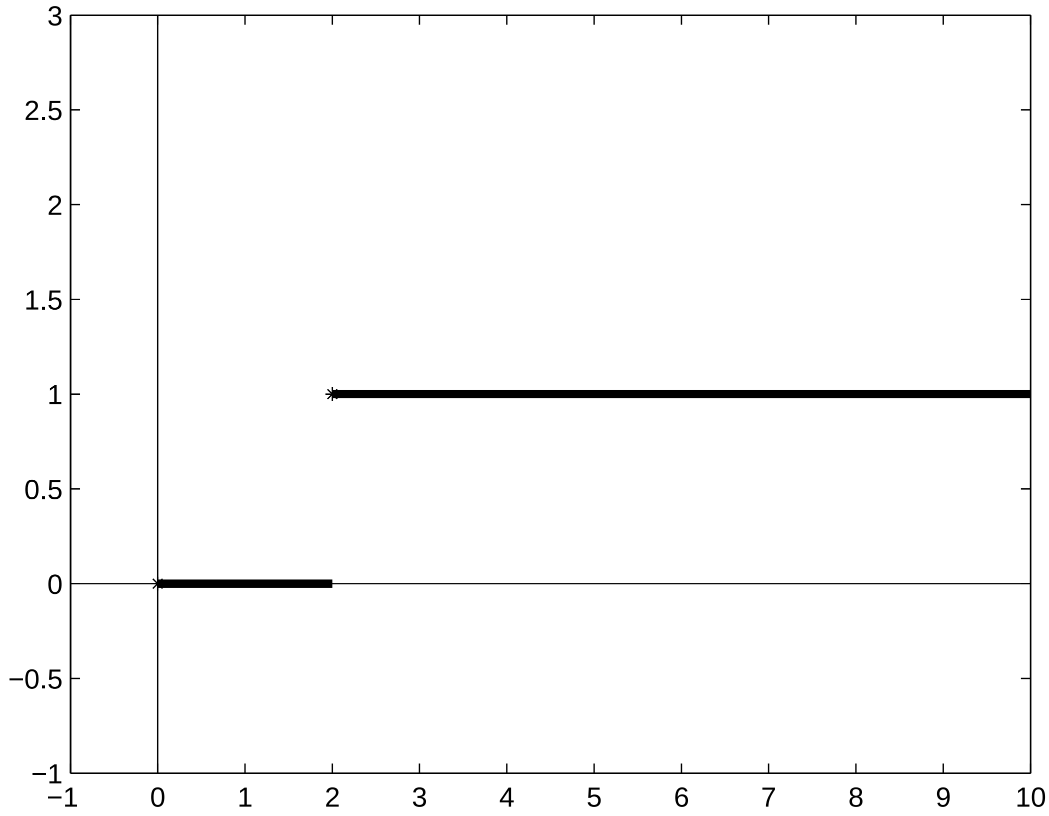
- Heaviside function, $u_c(t)$, and $\mathcal{L}\{u_c(t)\}$
- Other functions defined in terms of $u_c(t)$ and their transforms
- Dirac delta and its transform.

Heaviside Function

Unit step (Heaviside) function for $c \geq 0$

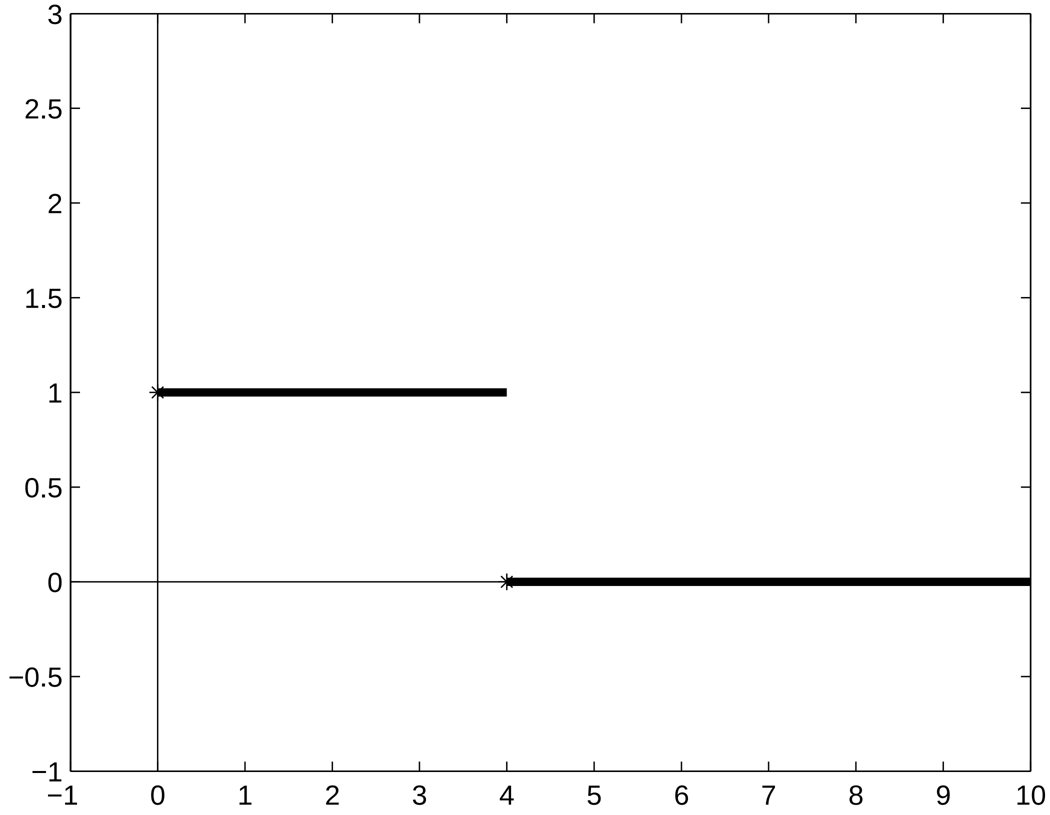
$$u_c(t) = \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases}$$

Heaviside Function



$u_2(t)$ for $t \geq 0$

Step Down via Heaviside

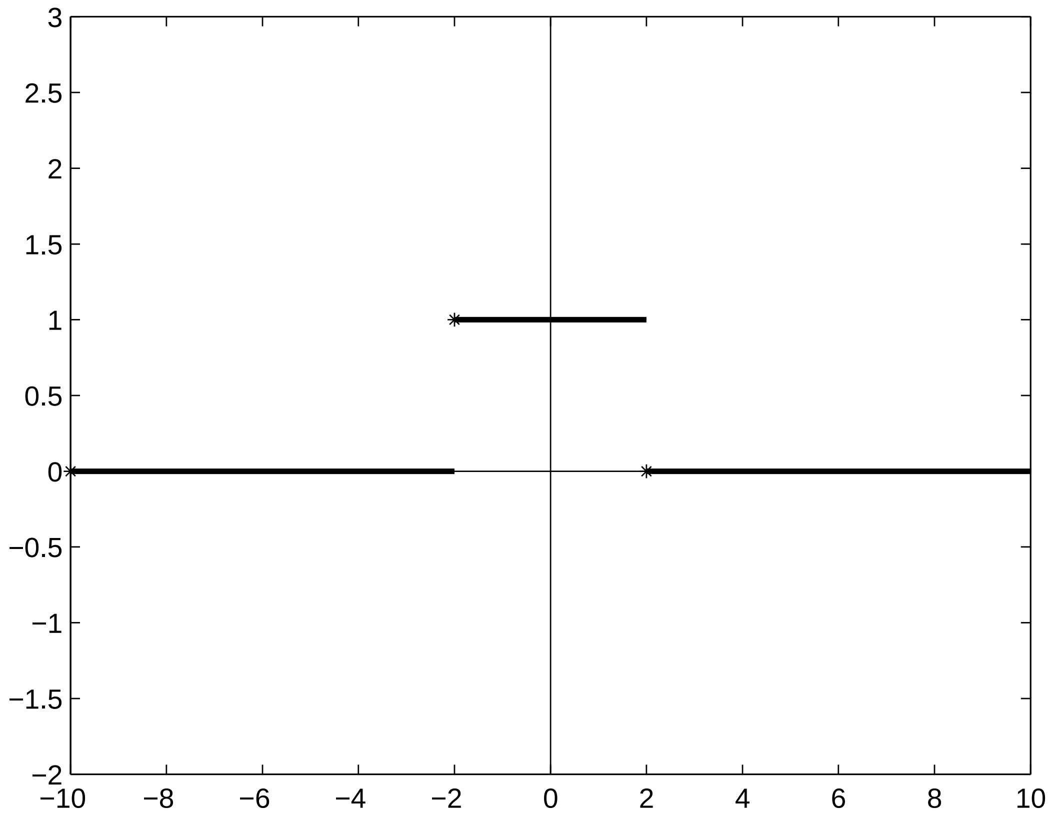


$$1 - u_4(t) \text{ for } t \geq 0$$

Pulse Function

$$f(t) = \begin{cases} \gamma & -\tau \leq t \leq \tau \\ 0 & \text{otherwise} \end{cases}$$

Pulse Function



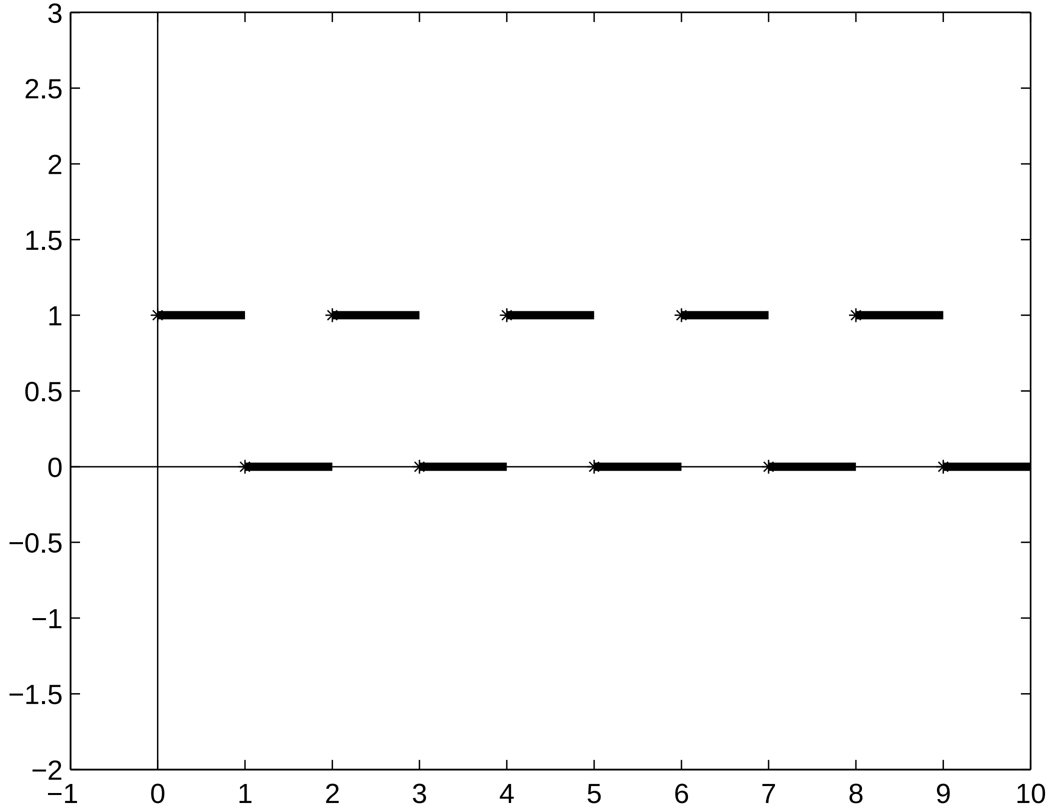
Pulse with $\gamma = 1$ and $\tau = 2$

Square Wave

Simple square wave, e.g., 1 and 0 alternately infinitely each with width 1.

$$f(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & 1 \le t < 2 \\ 1 & 2 \le t < 3 \\ \vdots & \vdots \end{cases}$$

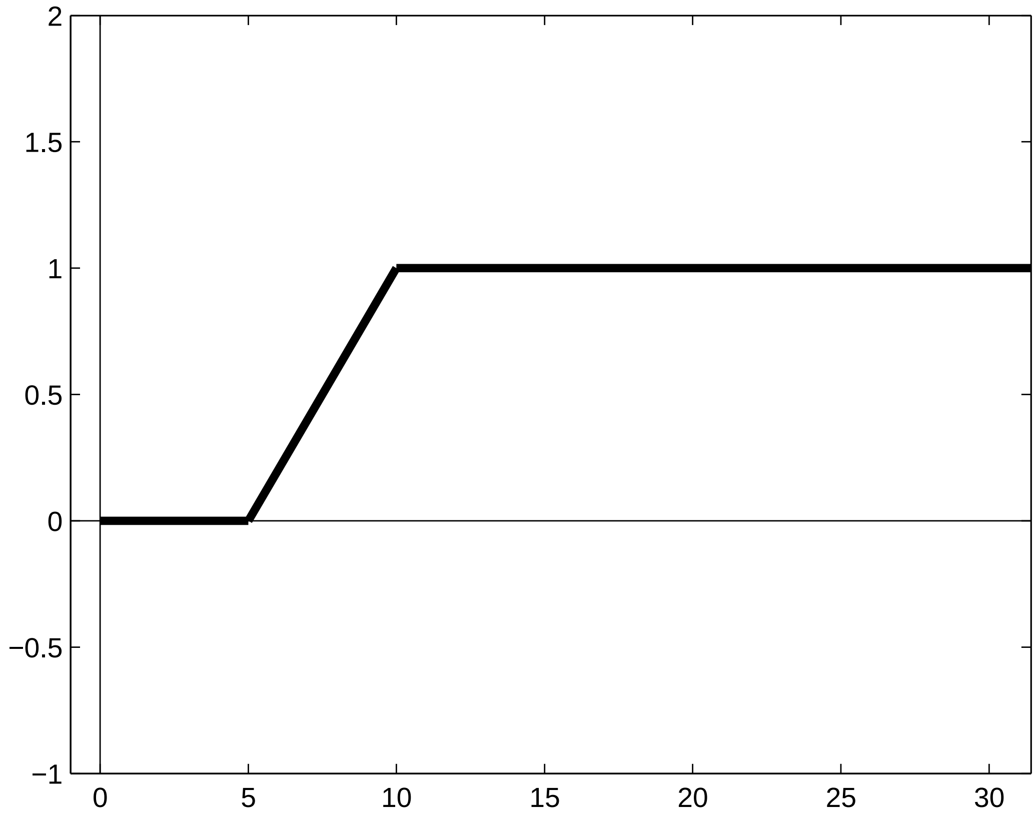
Square Wave



Ramp Loading

$$f(t) = \begin{cases} 0 & 0 \leq t < 5 \\ (t-5)/5 & 5 \leq t < 10 \\ 1 & t \geq 10 \\ \vdots & \vdots \end{cases}$$

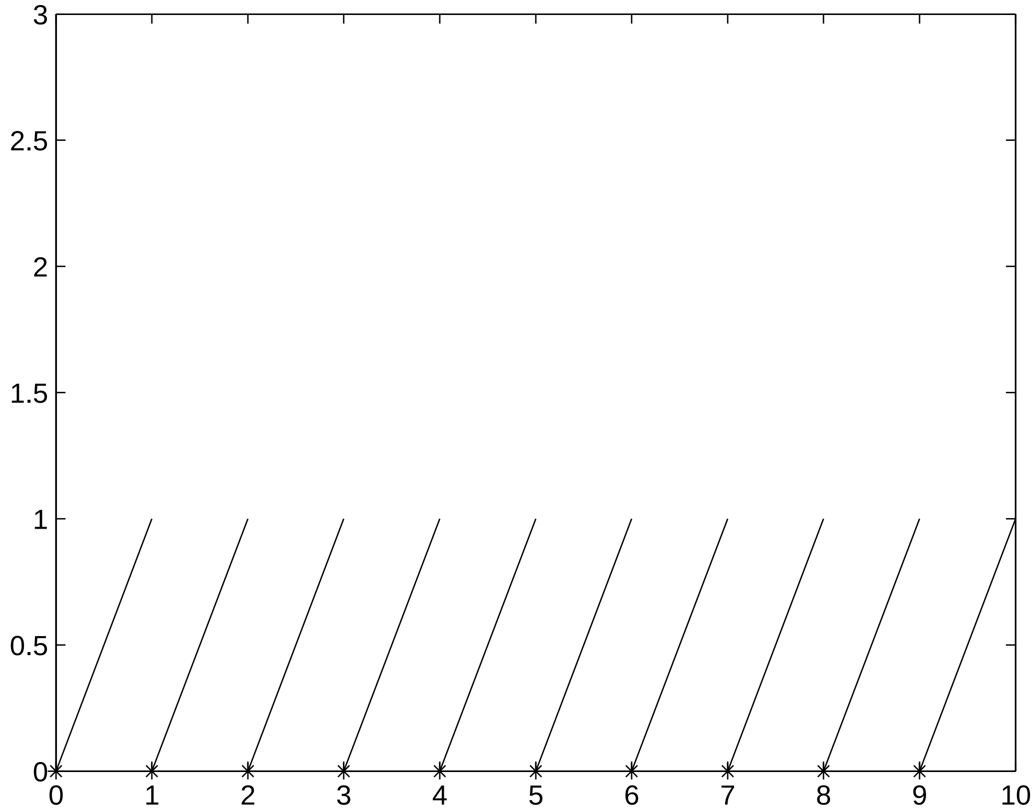
Ramp Loading



Saw Tooth Wave

$$f(t) = \begin{cases} t & 0 \leq t \leq 1 \\ f(t-1) & t > 1 \end{cases}$$

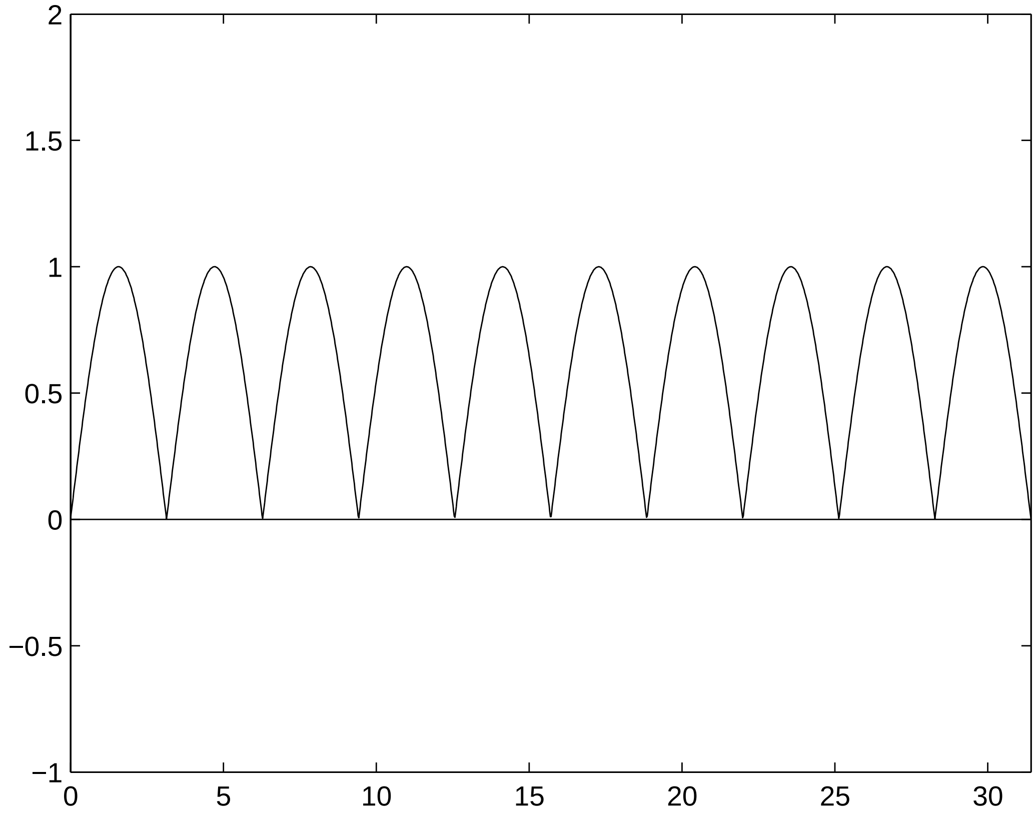
Saw Tooth Wave



Rectified Sine Wave

$$f(t) = \begin{cases} \sin t & 0 \le t \le \pi \\ f(t-\pi) & t > 1 \end{cases}$$

Rectified Sine Wave



Heaviside Function

Heaviside function for $c \geq 0$ is a unit step up at $t = c \geq 0$

$$u_c(t) = \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases}$$

$1 - u_c(t)$ gives a step down at $t = c$

Heaviside Function

The Laplace Transform follows easily from the definition:

$$\mathcal{L}\{u_c(t)\} = \int_0^\infty e^{-st} u_c(t)\, dt$$

$$= \int_c^\infty e^{-st} u_c(t)\, dt = -\lim_{A\to\infty} \Big[\frac{e^{-st}}{s}\Big]_c^A$$

$$= \frac{e^{-sc}}{s}, \quad s > 0$$

Heaviside Function

$u_c(t)$ can be used to translate a function

$$g(t) = \begin{cases} 0 & t < c \\ f(t-c) & t \geq c \end{cases} \Leftrightarrow g(t) = u_c(t) f(t-c)$$

$$\mathcal{L}\{u_c(t) f(t-c)\} = e^{-cs} \mathcal{L}\{f(t)\} = e^{-cs} F(s)$$

$$u_c(t) f(t-c) = \mathcal{L}^{-1}\{e^{-cs} F(s)\}$$

Translate $f(t)$ by $c \Leftrightarrow$ scale $F(s)$ by e^{-cs}

Remember $g(t) = 0$ for $t < c$.

Heaviside Function

It works in the other direction as well:

$$\mathcal{L}\{e^{ct}f(t)\} = F(s-c)$$

$$e^{ct}f(t) = \mathcal{L}^{-1}\{F(s-c)\}$$

Translate $F(s)$ by $c \Leftrightarrow$ scale $f(t)$ by e^{ct}

Examples

$$\mathcal{L}\{u_5(t)(t-5)\} = \frac{e^{-5s}}{s^2}$$

$$\mathcal{L}\{u_{10}(t)(t-10)\} = \frac{e^{-10s}}{s^2}$$

$$\mathcal{L}\{u_{\pi/4}(t)\cos(t-\pi/4)\} = e^{-\pi s/4}\frac{s}{s^2+1}$$

- $y = t$ shifted by 5 for $t \geq 5$
- $y = t$ shifted by 10 for $t \geq 10$
- $y = \cos t$ shifted by $\pi/4$ for $t \geq \pi/4$

Heaviside Function

$u_c(t)$ can be used to define a pulse of width Δ and height γ at $t = c$

$$f(t) = \begin{cases} \gamma & c \leq t \leq c + \Delta \\ 0 & \text{otherwise} \end{cases}$$

$$f(t) = \gamma u_c(t) - \gamma u_{c+\Delta}(t)$$

$$F(s) = \gamma \mathcal{L}\{u_c(t)\} - \gamma \mathcal{L}\{u_{c+\Delta}(t)\} = \gamma \frac{e^{-sc}}{s} - \gamma \frac{e^{-s(c+\Delta)}}{s}$$

$$= \gamma \frac{e^{-sc}}{s} \left[1 - e^{-s\Delta}\right]$$

Heaviside Function

$u_c(t)$ can be used to define square waves:

Square wave, e.g., 1 and 0 alternately on $t \geq 0$ (finite or infinite)

$$f(t) = 1 - u_1(t) + u_2(t) - u_3(t) \quad 1,\ 0,\ 1,\ 0,\ 0\ldots$$

$$f(t) = 1 + \sum_{k=1}^{\infty} (-1)^k u_k(t) \quad 1,\ 0 \text{ alternating infinitely}$$

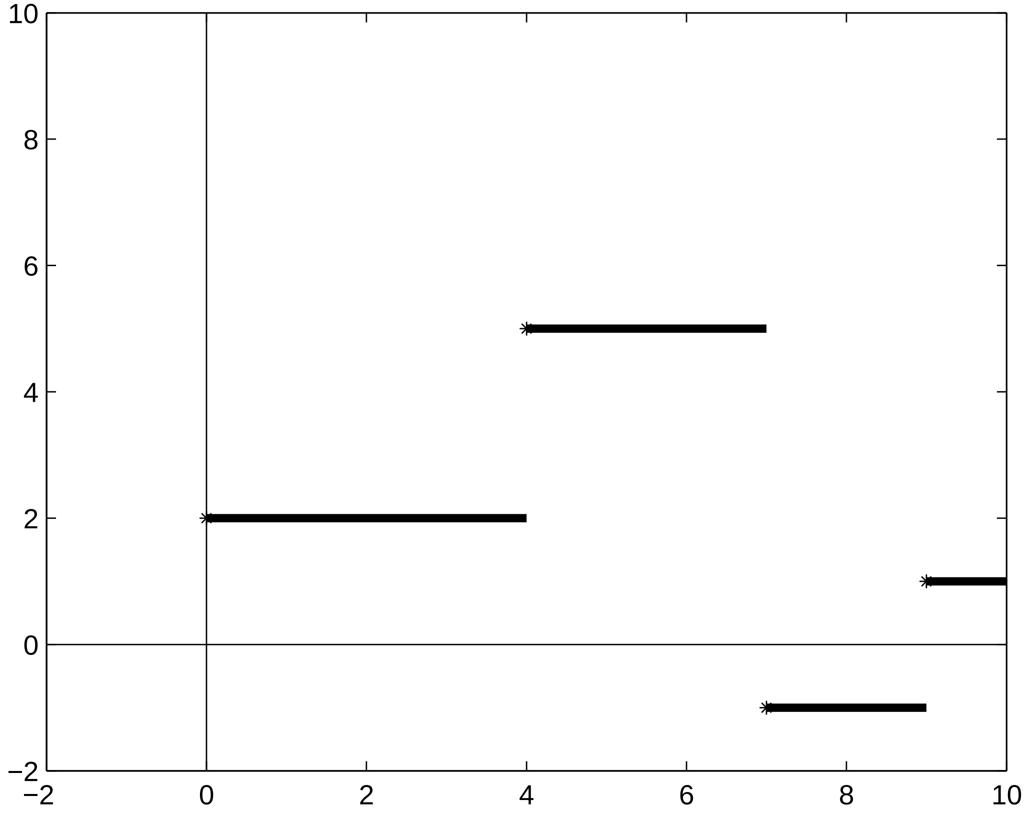
Heaviside Function

$u_c(t)$ can be used to define more complicated discontinuous functions:

$$f(t) = \begin{cases} 2 & 0 \le t < 4 \\ 5 & 4 \le t < 7 \\ -1 & 7 \le t < 9 \\ 1 & t \ge 9 \end{cases} \Leftrightarrow f(t) = 2 + 3u_4(t) - 6u_7(t) + 2u_9(t)$$

$$f(t) = \begin{cases} 2 & 0 \le t < 4 \\ 2+3 & 4 \le t < 7 \\ 2+3-6 & 7 \le t < 9 \\ 2+3-6+2 & t \ge 9 \end{cases}$$

Example



Heaviside Function

$$f(t) = 2 + 3u_4(t) - 6u_7(t) + 2u_9(t)$$

$$F(s) = 2\mathcal{L}\{1\} + 3\mathcal{L}\{u_4(t)\} - 6\mathcal{L}\{u_7(t)\} + 2\mathcal{L}\{u_9(t)\}$$

$$= \frac{2}{s} + \frac{3e^{-4s}}{s} - \frac{6e^{-7s}}{s} + \frac{2e^{-9s}}{s}$$

$$= \frac{1}{s}\left[2 + 3e^{-4s} - 6e^{-7s} + 2e^{-9s}\right]$$

Example

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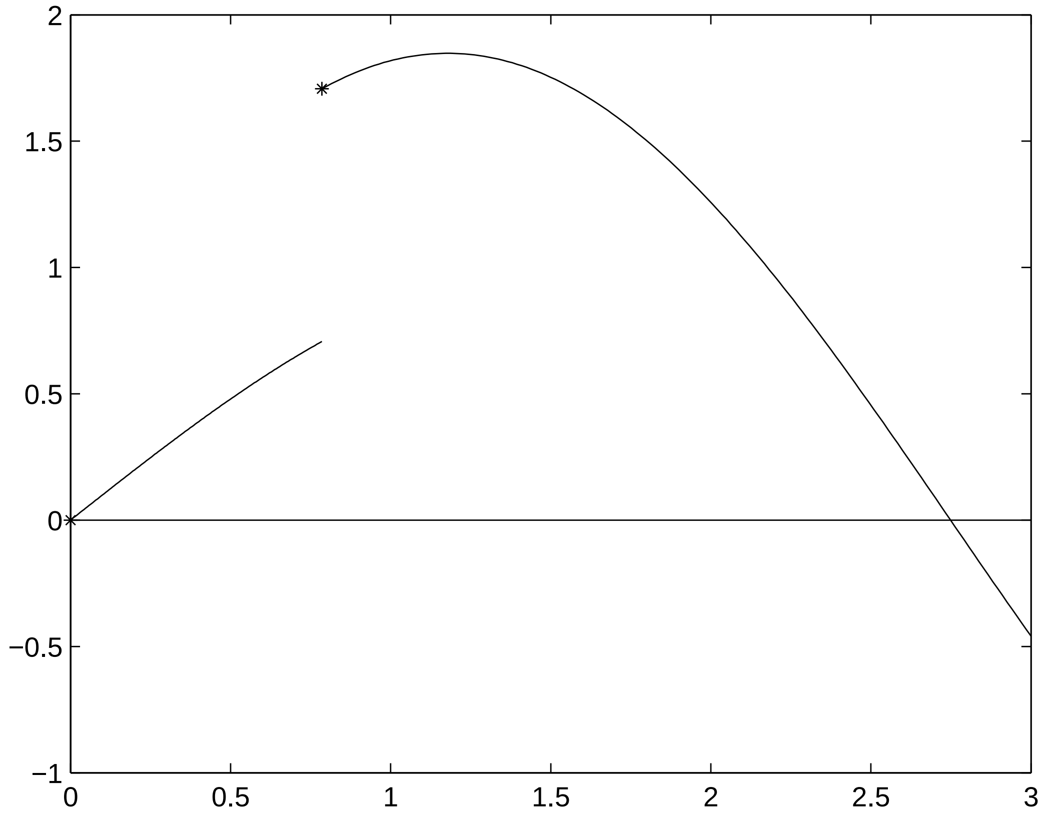
$$f(t) = \begin{cases} \sin t & 0 \le t < \frac{\pi}{4} \\ \sin t + \cos(t - \frac{\pi}{4}) & t \ge \frac{\pi}{4} \end{cases}$$

$$f(t) = \sin t + g(t)$$

$$g(t) = \begin{cases} 0 & 0 \le t < \frac{\pi}{4} \\ \cos(t - \frac{\pi}{4}) & t \ge \frac{\pi}{4} \end{cases} = u_{\pi/4}(t)\cos(t - \frac{\pi}{4})$$

Example



Example

Compute the Laplace Transform:

$$f(t) = \sin t + u_{\pi/4}(t)\cos(t - \frac{\pi}{4})$$

$$F(s) = \mathcal{L}\{\sin t\} + \mathcal{L}\{u_{\pi/4}(t)\cos(t - \frac{\pi}{4})\}$$

$$= \frac{1}{s^2+1} + e^{-\pi s/4}\mathcal{L}\{\cos t\}$$

$$= \frac{1}{s^2+1} + e^{-\pi s/4}\frac{s}{s^2+1}$$

Example

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$$G(s) = \frac{1}{s^2 - 4s + 5}$$

complete the square

$$G(s) = \frac{1}{(s-2)^2 + 1} = F(s-2)$$

$$F(s) = \frac{1}{(s^2+1)} \rightarrow f(t) = \sin t$$

$$\therefore g(t) = e^{2t} \sin t$$

Impulse of a Forcing Function

Definition 17.1. Let $f(t)$ be a forcing function then the impulse, $\mathcal{I}(\tau)$, around $t = t_0$ of $f(t)$ is

$$\mathcal{I}(\tau) = \int_{t_0-\tau}^{t_0+\tau} f(t)dt$$

If $f(t)$ is a pulse with height γ whose nonzero value interval corresponds to $t_0 - \tau \leq t \leq t_0 + \tau$ then $I(\tau) = 2\gamma\tau$.

Impulse of a Forcing Function

The pulse function

$$d_\tau(t) = \begin{cases} \frac{1}{2\tau} & -\tau < t < \tau \\ 0 & t \leq -\tau \text{ or } t \geq \tau \end{cases}$$

has an impulse of 1, i.e., $I(\tau) = 1$, for any $\tau \neq 0$.

Lemma 17.1. *If* $t \neq 0$ *then*

$$\lim_{\tau \to 0} d_\tau(t) = 0$$

Dirac Delta Function

Since $I(\tau) = 1$, for any $\tau \neq 0$ we have

$$\lim_{\tau \to 0} I(\tau) = 1$$

Definition 17.2. A unit impulse function or Dirac delta function is a generalized function defined by:

$$\delta(t - t_0) = 0, \quad t \neq t_0$$

$$\int_{-\infty}^{\infty} \delta(t - t_0)\, dt = 1$$

Dirac Delta Function

Definition 17.3. The Laplace Transform of the Dirac delta function is defined to be:

$$\mathcal{L}\{\delta(t - t_0)\} = e^{-st_0}$$

$$\mathcal{L}\{\delta(t)\} = 1$$

Furthermore, using the Dirac delta function as a weight in integration is defined to have the result:

$$\int_{-\infty}^{\infty} \delta(t - t_0) f(t) \; dt = f(t_0)$$