

Set 18: Laplace Transform and IVPs Part 3

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Ordinary Differential Equations

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General Form

$$ay'' + by' + cy = f(t), y(0) = y_0, \quad y'(0) = y'_0$$

\Leftrightarrow

$$Y(s) = \frac{(as + b)y_0 + ay'_0}{as^2 + bs + c} + \frac{F(s)}{as^2 + bs + c}$$

where $F(s) = \mathcal{L}\{f\}$.

Example

Ramp loading:

- $f(t)$ continuous
- $f'(t)$ discontinuous at two points $t = 5$ and $t = 10$

$$y'' + 4y = f(t)$$

$$y(0) = 0$$

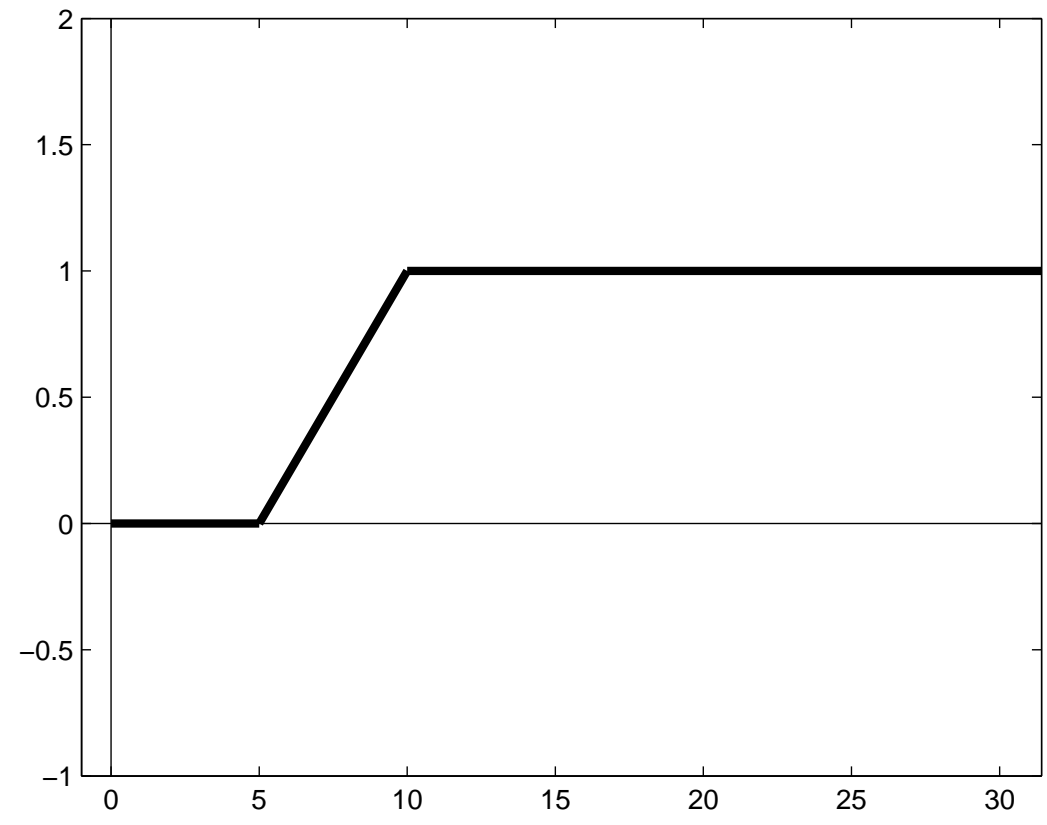
$$y'(0) = 0$$

We could solve it using our standard techniques in a piecewise manner.

Ramp Loading

$$f(t) = \begin{cases} 0 & 0 \leq t < 5 \\ (t - 5)/5 & 5 \leq t < 10 \\ 1 & t \geq 10 \\ \vdots & \vdots \end{cases}$$

Ramp Loading



Example

$$y'' + 4y = f(t), \quad y(0) = 0, \quad y'(0) = 0$$

$$a = 1, \quad b = 0, \quad c = 4$$

$$Y(s) = \frac{0}{s^2 + 4} + \frac{F(s)}{s^2 + 4}$$

$$f(t) = \frac{1}{5} (u_5(t)(t - 5) - u_{10}(t)(t - 10))$$

$$F(s) = \frac{1}{5} \left[\frac{e^{-5s}}{s^2} - \frac{e^{-10s}}{s^2} \right]$$

Example

$$\begin{aligned} Y(s) &= \frac{1}{5} \left[\frac{1}{s^2 + 4} \right] \left[\frac{e^{-5s} - e^{-10s}}{s^2} \right] \\ &= \frac{e^{-5s} - e^{-10s}}{5} \left[\frac{1}{s^2(s^2 + 4)} \right] = \frac{e^{-5s} - e^{-10s}}{5} G(s) \end{aligned}$$

$$G(s) = \left[\frac{1}{s^2(s^2 + 4)} \right]$$

Example

$$G(s) = \left[\frac{1}{s^2(s^2 + 4)} \right] = \frac{c_1}{s^2} + \frac{c_2}{s^2 + 4} = \frac{(c_1 + c_2)s^2 + 4c_1}{s^2(s^2 + 4)}$$

$$\begin{aligned} G(s) &= \frac{1/4}{s^2} - \frac{1/4}{s^2 + 4} \\ &= \frac{1}{4} \times \frac{1}{s^2} - \frac{1}{8} \times \frac{2}{s^2 + 4} \end{aligned}$$

$$\therefore g(t) = \frac{1}{4}t - \frac{1}{8} \sin 2t$$

Example

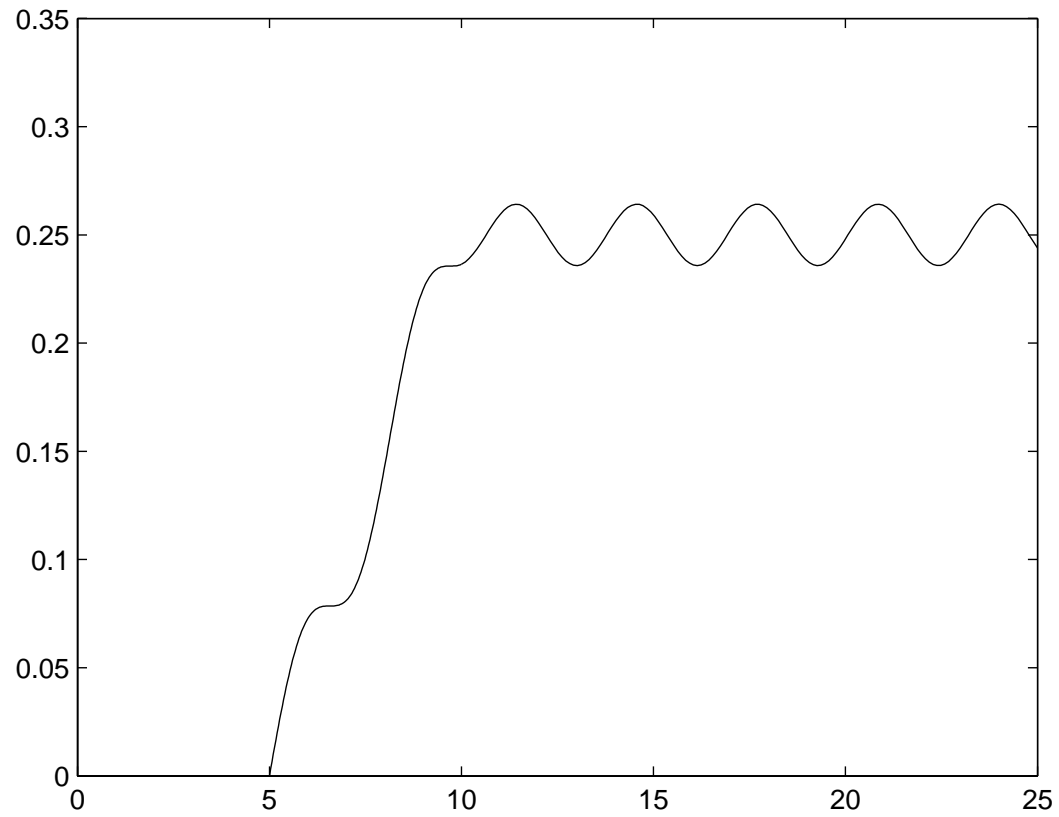
$$g(t) = \frac{1}{4}t - \frac{1}{8} \sin 2t \quad \text{and} \quad Y(s) = \frac{e^{-5s} - e^{-10s}}{5} G(s)$$

⇓

$$y(t) = \frac{1}{5} [u_5(t)g(t-5) - u_{10}(t)g(t-10)]$$

$$= \begin{cases} 0 & 0 \leq t \leq 5 \\ 0.20 [0.25(t-5) - 0.125 \sin 2(t-5)] & 5 \leq t \leq 10 \\ 0.25 - 0.025(\sin 2(t-5) + \sin 2(t-10)) & t \geq 10 \end{cases}$$

Ramp Loading Solution



Example

- $f(t)$ continuous
- $f'(t)$ has discontinuities at $t = 5$ and $t = 10$
- $y(t), y'(t), y''(t)$ continuous
- $y'''(t)$ has discontinuities at $t = 5$ and $t = 10$

Discontinuities

Given

$$y'' + p(t)y' + q(t)y = f(t)$$

If $p(t)$ and $q(t)$ are continuous, and $f(t)$ is piecewise continuous on $\alpha < t < \beta$ then

- $y(t)$ and $y'(t)$ are continuous on $\alpha < t < \beta$
- $y''(t)$ is discontinuous with discontinuities at the same points as $f(t)$.

Example

Pulse forcing function:

$$f(t) = u_5(t) - u_{20}(t) = \begin{cases} 1 & 5 \leq t < 20 \\ 0 & 0 \leq t < 5 \text{ and } t \geq 20 \end{cases}$$

Example

$$2y'' + y' + 2y = f(t), \quad y(0) = 0, \quad y'(0) = 0$$

$$a = 2, \quad b = 1, \quad c = 2$$

$$Y(s) = \frac{F(s)}{(2s^2 + s + 2)} + \frac{(2s + 1)y(0) + 2y'(0)}{(2s^2 + s + 2)}$$

$$Y(s) = \frac{F(s)}{(2s^2 + s + 2)} = \frac{e^{-5s} - e^{-20s}}{s(2s^2 + s + 2)}$$

$$= (e^{-5s} - e^{-20s}) \frac{1}{s(2s^2 + s + 2)} = (e^{-5s} - e^{-20s})G(s)$$

Example

$$\begin{aligned} G(s) &= \frac{1}{s(2s^2 + s + 2)} \\ &= \frac{c_1}{s} + \frac{c_2s + c_3}{2s^2 + s + 2} \end{aligned}$$

$$2c_1s^2 + c_1s + 2c_1 + c_2s^2 + c_3s = (2c_1 + c_2)s^2 + (c_1 + c_3)s + 2c_1$$

$$2c_1 = 1 \rightarrow c_1 = 1/2$$

$$2c_1 + c_2 = 0 \rightarrow c_2 = -1$$

$$c_1 + c_3 = 0 \rightarrow c_3 = -1/2$$

$$G(s) = \frac{1}{2s} - \frac{s + 1/2}{2s^2 + s + 2}$$

Example

$$\begin{aligned} G(s) &= \frac{1}{2s} - \frac{s + 1/2}{2s^2 + s + 2} = \frac{1}{2s} - \frac{1}{2} \frac{s + 1/2}{(s^2 + 0.5s + 1)} \\ &= \frac{1}{2s} - \frac{1}{2} \frac{s + 1/2}{(s + 0.25)^2 + 1 - 1/16} = \frac{1}{2s} - \frac{1}{2} \frac{(s + 0.25) + 0.25}{(s + 0.25)^2 + 15/16} \\ &= \frac{1}{2s} - \frac{1}{2} \frac{(s + 0.25)}{(s + 0.25)^2 + 15/16} - \frac{1}{2} \frac{0.25}{(s + 0.25)^2 + 15/16} \end{aligned}$$

$$\text{recall } \mathcal{L}\{e^{at} \cos bt\} = \frac{s - a}{(s - a)^2 + b^2}$$

$$\mathcal{L}\{e^{at} \sin bt\} = \frac{b}{(s - a)^2 + b^2}$$

Example

$$\mathcal{L}^{-1} \frac{1}{2} \frac{(s + 0.25)}{(s + 0.25)^2 + 15/16} = 0.5e^{-0.25t} \cos(0.25\sqrt{15}t)$$

$$\frac{1}{2} \frac{0.25}{(s + 0.25)^2 + 15/16} = \frac{1}{2\sqrt{15}} \frac{0.25\sqrt{15}}{(s + 0.25)^2 + 15/16}$$

$$\begin{aligned} \mathcal{L}^{-1} \frac{1}{2\sqrt{15}} \frac{0.25\sqrt{15}}{(s + 0.25)^2 + 15/16} &= \frac{1}{2\sqrt{15}} e^{-0.25t} \sin(0.25\sqrt{15}t) \\ &= \frac{\sqrt{15}}{30} e^{-0.25t} \sin(0.25\sqrt{15}t) \end{aligned}$$

Example

$$Y(s) = (e^{-5s} - e^{-20s})G(s)$$

$$G(s) = \frac{1}{2s} - \frac{1}{2} \frac{(s + 0.25)}{(s + 0.25)^2 + 15/16} - \frac{1}{2\sqrt{15}} \frac{0.25\sqrt{15}}{(s + 0.25)^2 + 15/16}$$

$$g(t) = 0.5 - 0.5e^{-0.25t} \cos(0.25\sqrt{15}t) - \frac{\sqrt{15}}{30} e^{-0.25t} \sin(0.25\sqrt{15}t)$$

$$y(t) = u_5(t)g(t - 5) - u_{20}(t)g(t - 20)$$

Example

$$2y'' + y' + 2y = \delta(t - 5), \quad y(0) = 0, \quad y'(0) = 0$$

$$a = 2, \quad b = 1, \quad c = 2$$

$$\begin{aligned} Y(s) &= \frac{F(s)}{(2s^2 + s + 2)} + \frac{(2s + 1)y(0) + 2y'(0)}{(2s^2 + s + 2)} \\ &= \frac{e^{-5s}}{2} \frac{1}{(s^2 + 0.5s + 1)} = \frac{e^{-5s}}{2} \frac{1}{(s + 0.25)^2 + 15/16} \\ &= \frac{e^{-5s}}{2} \frac{4}{\sqrt{15}} \frac{\sqrt{15}/4}{(s + 0.25)^2 + 15/16} \end{aligned}$$

Example

$$Y(s) = \frac{e^{-5s}}{2} \frac{4}{\sqrt{15}} \frac{\sqrt{15}/4}{(s + 0.25)^2 + 15/16}$$

$$\mathcal{L}^{-1}\left\{\frac{4}{\sqrt{15}} \frac{\sqrt{15}/4}{(s + 0.25)^2 + 15/16}\right\} = \frac{4}{\sqrt{15}} e^{-0.25t} \sin 0.25\sqrt{15}t$$

$$y(t) = 0.5u_5(t) \mathcal{L}^{-1}\left\{\frac{4}{\sqrt{15}} \frac{\sqrt{15}/4}{(s + 0.25)^2 + 15/16}\right\}(t - 5)$$

$$y(t) = \frac{2}{\sqrt{15}} u_5(t) e^{-0.25(t-5)} \sin 0.25\sqrt{15}(t - 5)$$