Set 18: Laplace Transform and IVPs Part 3

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Ordinary Differential Equations
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General Form

$$ay'' + by' + cy = f(t), y(0) = y_0, \quad y'(0) = y'_0$$

$$\Updownarrow$$

$$Y(s) = \frac{(as+b)y_0 + ay'_0}{as^2+bs+c} + \frac{F(s)}{as^2+bs+c}$$

where $F(s) = \mathcal{L}\{f\}$.

Example

Ramp loading:

- $f(t)$ continuous
- $f'(t)$ discontinuous at two points $t = 5$ and $t = 10$

$$y'' + 4y = f(t)$$

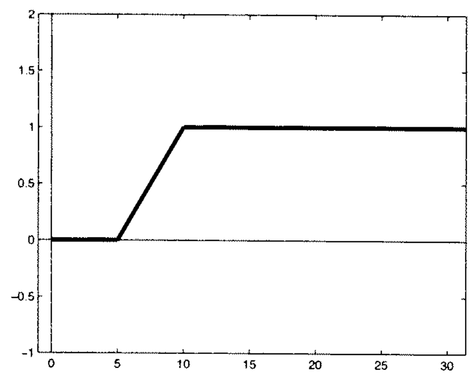
$$y(0) = 0$$
$$y'(0) = 0$$

We could solve it using our standard techniques in a piecewise manner.

Ramp Loading

$$f(t) = \begin{cases} 0 & 0 \le t < 5 \\ (t-5)/5 & 5 \le t < 10 \\ 1 & t \ge 10 \\ \vdots & \vdots \end{cases}$$

Ramp Loading



Example

$$y'' + 4y = f(t), \quad y(0) = 0, \quad y'(0) = 0$$
$$a = 1, \quad b = 0, \quad c = 4$$

$$Y(s) = \frac{0}{s^2+4} + \frac{F(s)}{s^2+4}$$

$$f(t) = \frac{1}{5}\big(u_5(t)(t-5) - u_{10}(t)(t-10)\big)$$
$$F(s) = \frac{1}{5}\Big[\frac{e^{-5s}}{s^2} - \frac{e^{-10s}}{s^2}\Big]$$

Example

$$Y(s) = \frac{1}{5}\Big[\frac{1}{s^2+4}\Big]\Big[\frac{e^{-5s} - e^{-10s}}{s^2}\Big]$$
$$= \frac{e^{-5s} - e^{-10s}}{5}\Big[\frac{1}{s^2(s^2+4)}\Big] = \frac{e^{-5s} - e^{-10s}}{5} G(s)$$

$$G(s) = \Big[\frac{1}{s^2(s^2+4)}\Big]$$

Example

$$G(s) = \Big[\frac{1}{s^2(s^2+4)}\Big] = \frac{c_1}{s^2} + \frac{c_2}{s^2+4} = \frac{(c_1+c_2)s^2 + 4c_1}{s^2(s^2+4)}$$

$$G(s) = \frac{1/4}{s^2} - \frac{1/4}{s^2+4}$$
$$= \frac{1}{4} \times \frac{1}{s^2} - \frac{1}{8} \times \frac{2}{s^2+4}$$

$$\therefore g(t) = \frac{1}{4}t - \frac{1}{8}\sin 2t$$

Example

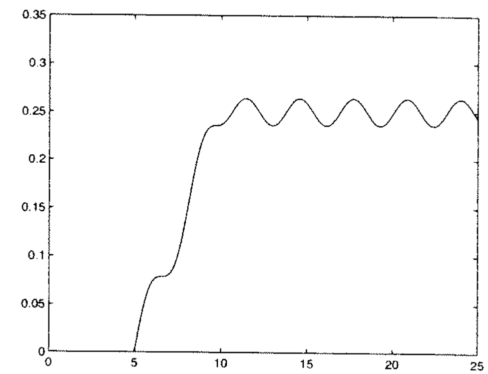
$$g(t) = \frac{1}{4}t - \frac{1}{8}\sin 2t \quad \text{and} \quad Y(s) = \frac{e^{-5s} - e^{-10s}}{5}G(s)$$

$$\Downarrow$$

$$y(t) = \frac{1}{5}\left[u_5(t)g(t-5) - u_{10}(t)g(t-10)\right]$$

$$= \begin{cases} 0 & 0 \le t \le 5 \\ 0.20\left[0.25(t-5) - 0.125\sin 2(t-5)\right] & 5 \le t \le 10 \\ 0.25 - 0.025(\sin 2(t-5) + \sin 2(t-10)) & t \ge 10 \end{cases}$$

Ramp Loading Solution



Example

- $f(t)$ continuous
- $f'(t)$ has discontinuities at $t = 5$ and $t = 10$
- $y(t), y'(t), y''(t)$ continuous
- $y'''(t)$ has discontinuities at $t = 5$ and $t = 10$

Discontinuities

Given

$$y'' + p(t)y' + q(t)y = f(t)$$

If $p(t)$ and $q(t)$ are continuous, and $f(t)$ is piecewise continuous on $\alpha < t < \beta$ then

- $y(t)$ and $y'(t)$ are continuous on $\alpha < t < \beta$
- $y''(t)$ is discontinuous with discontinuities at the same points as $f(t)$.

Example

Pulse forcing function:

$$f(t) = u_5(t) - u_{20}(t) = \begin{cases} 1 & 5 \le t < 20 \\ 0 & 0 \le t < 5 \text{ and } t \ge 20 \end{cases}$$

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Example

$$2y'' + y' + 2y = f(t), \quad y(0) = 0, \quad y'(0) = 0$$
$$a = 2, \quad b = 1, \quad c = 2$$

$$Y(s) = \frac{F(s)}{(2s^2+s+2)} + \frac{(2s+1)y(0) + 2y'(0)}{(2s^2+s+2)}$$
$$Y(s) = \frac{F(s)}{(2s^2+s+2)} = \frac{e^{-5s} - e^{-20s}}{s(2s^2+s+2)}$$
$$= (e^{-5s} - e^{-20s})\frac{1}{s(2s^2+s+2)} = (e^{-5s} - e^{-20s})G(s)$$

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Example

$$G(s) = \frac{1}{s(2s^2+s+2)}$$
$$= \frac{c_1}{s} + \frac{c_2 s + c_3}{2s^2+s+2}$$

$$2c_1 s^2 + c_1 s + 2c_1 + c_2 s^2 + c_3 s = (2c_1 + c_2)s^2 + (c_1 + c_3)s + 2c_1$$
$$2c_1 = 1 \rightarrow c_1 = 1/2$$
$$2c_1 + c_2 = 0 \rightarrow c_2 = -1$$
$$c_1 + c_3 = 0 \rightarrow c_3 = -1/2$$

$$G(s) = \frac{1}{2s} - \frac{s+1/2}{2s^2+s+2}$$

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Example

$$G(s) = \frac{1}{2s} - \frac{s+1/2}{2s^2+s+2} = \frac{1}{2s} - \frac{1}{2}\frac{s+1/2}{(s^2+0.5s+1)}$$
$$= \frac{1}{2s} - \frac{1}{2}\frac{s+1/2}{(s+0.25)^2+1-1/16} = \frac{1}{2s} - \frac{1}{2}\frac{(s+0.25)+0.25}{(s+0.25)^2+15/16}$$
$$= \frac{1}{2s} - \frac{1}{2}\frac{(s+0.25)}{(s+0.25)^2+15/16} - \frac{1}{2}\frac{0.25}{(s+0.25)^2+15/16}$$

$$\text{recall} \quad \mathcal{L}\{e^{at}\cos bt\} = \frac{s-a}{(s-a)^2+b^2}$$
$$\mathcal{L}\{e^{at}\sin bt\} = \frac{b}{(s-a)^2+b^2}$$

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Example

$$\mathcal{L}^{-1}\frac{1}{2}\frac{(s+0.25)}{(s+0.25)^2+15/16} = 0.5e^{-0.25t}\cos(0.25\sqrt{15}t)$$

$$\frac{1}{2}\frac{0.25}{(s+0.25)^2+15/16} = \frac{1}{2\sqrt{15}}\frac{0.25\sqrt{15}}{(s+0.25)^2+15/16}$$

$$\mathcal{L}^{-1}\frac{1}{2\sqrt{15}}\frac{0.25\sqrt{15}}{(s+0.25)^2+15/16} = \frac{1}{2\sqrt{15}}e^{-0.25t}\sin(0.25\sqrt{15}t)$$

$$= \frac{\sqrt{15}}{30}e^{-0.25t}\sin(0.25\sqrt{15}t)$$

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Example

$$Y(s) = (e^{-5s} - e^{-20s})G(s)$$

$$G(s) = \frac{1}{2s} - \frac{1}{2}\frac{(s+0.25)}{(s+0.25)^2+15/16} - \frac{1}{2\sqrt{15}}\frac{0.25\sqrt{15}}{(s+0.25)^2+15/16}$$

$$g(t) = 0.5 - 0.5e^{-0.25t}\cos(0.25\sqrt{15}t) - \frac{\sqrt{15}}{30}e^{-0.25t}\sin(0.25\sqrt{15}t)$$

$$y(t) = u_5(t)g(t-5) - u_{20}(t)g(t-20)$$

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Example

$$2y'' + y' + 2y = \delta(t-5), \quad y(0) = 0, \quad y'(0) = 0$$

$$a = 2, \quad b = 1, \quad c = 2$$

$$Y(s) = \frac{F(s)}{(2s^2+s+2)} + \frac{(2s+1)y(0) + 2y'(0)}{(2s^2+s+2)}$$

$$= \frac{e^{-5s}}{2}\frac{1}{(s^2+0.5s+1)} = \frac{e^{-5s}}{2}\frac{1}{(s+0.25)^2+15/16}$$

$$= \frac{e^{-5s}}{2}\frac{4}{\sqrt{15}}\frac{\sqrt{15}/4}{(s+0.25)^2+15/16}$$

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Example

$$Y(s) = \frac{e^{-5s}}{2}\frac{4}{\sqrt{15}}\frac{\sqrt{15}/4}{(s+0.25)^2+15/16}$$

$$\mathcal{L}^{-1}\{\frac{4}{\sqrt{15}}\frac{\sqrt{15}/4}{(s+0.25)^2+15/16}\} = \frac{4}{\sqrt{15}}e^{-0.25t}\sin 0.25\sqrt{15}t$$

$$y(t) = 0.5u_5(t)\mathcal{L}^{-1}\{\frac{4}{\sqrt{15}}\frac{\sqrt{15}/4}{(s+0.25)^2+15/16}\}(t-5)$$

$$y(t) = \frac{2}{\sqrt{15}}u_5(t)e^{-0.25(t-5)}\sin 0.25\sqrt{15}(t-5)$$

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