Set 19: Laplace Transform and IVPs Part 4

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Ordinary Differential Equations

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Equivalence

Example of time domain and Laplace domain equivalent operations

Time Domain	Laplace Domain
linear combination	linear combination
differentiation	multiplication by s
multiplication by t	differentiation
translation by c	multiplication by e^{-cs}
multiplication by e^{ct}	translation by c

What about products of Laplace transforms?

Convolution Integral

Theorem 19.1. *Let $f(t)$ and $g(t)$ be two functions whose Laplace transforms are $F(s)$ and $G(s)$ respectively for $s > a$. If*

$$H(s) = F(s)G(s), \quad s > a$$

then

$$h(t) = \mathcal{L}^{-1}\{H(s)\} = \int_0^t f(t-\tau)g(\tau)d\,\tau = \int_0^t f(\tau)g(t-\tau)d\,\tau = f * g$$

General Form

$$ay'' + by' + cy = g(t), y(0) = y_0, \quad y'(0) = y'_0$$

$$\Updownarrow$$

$$Y(s) = \frac{(as+b)y_0 + ay'_0}{as^2 + bs + c} + \frac{G(s)}{as^2 + bs + c}$$

$$= \Phi(s) + \Psi(s) = \Phi(s) + H(s)G(s)$$

where $G(s) = \mathcal{L}\{g\}$ and $H(s) = (as^2 + bs + c)^{-1}$.

Another General Form

$$Y(s) = \Phi(s) + \Psi(s) = \Phi(s) + H(s)G(s)$$

$$y(t) = \phi(t) + \psi(t) = \phi(t) + h(t) * g(t)$$

- $\phi(t)$ is a solution to the homogeneous ODE that satisfies $y(0) = y_0, \quad y'(0) = y'_0$
- $\psi(t)$ is a particular solution that satisfies the "at rest" conditions $y(0) = y'(0) = 0$.

Another General Form

$$Y(s) = \Phi(s) + \Psi(s) = \Phi(s) + H(s)G(s)$$
$$y(t) = \phi(t) + \psi(t) = \phi(t) + h(t) * g(t)$$

- The transfer function $H(s) = (as^2 + bs + c)^{-1}$ depends only on the ODE, i.e., on the system modeled.
- It gives the response to a forcing function specified by $G(s)$ from an at rest condition.
- $G(s) = 1 \rightarrow g(t) = \delta(t)$ then
  $Y(s) = H(s) \rightarrow y(t) = h(t) * \delta(t) = h(t)$ for $t > 0$.
- The transfer function $H(s)$ is the Laplace transform of the impulse response $h(t)$.

Example

$$y'' + 4y = g(t), \quad y(0) = 3, \;\; y'(0) = -1$$

$$a = 1, \quad b = 0, \quad c = 4$$

$$Y(s) = \frac{(as+b)y_0 + ay_0'}{as^2 + bs + c} + \frac{G(s)}{as^2 + bs + c}$$

$$= \frac{3s - 1}{(s^2 + 4)} + \frac{G(s)}{(s^2 + 4)}$$

$$\Phi(s) = H(s)(3s - 1), \quad \Psi(s) = H(s)G(s), \quad H(s) = \frac{1}{(s^2 + 4)}$$

Example

Find $\phi(t)$:

$$\Phi(s) = \frac{(3s-1)}{(s^2+4)}$$

$$= 3\frac{s}{(s^2+4)} - \frac{1}{(s^2+4)}$$

$$= 3\frac{s}{(s^2+4)} - \frac{1}{2}\frac{2}{(s^2+4)}$$

$$\phi(t) = 3\cos 2t - \frac{1}{2}\sin 2t$$

Example

$$\phi(t) = 3\cos 2t - \frac{1}{2}\sin 2t$$

$$\phi'(t) = -6\sin 2t - \cos 2t$$

$$\phi''(t) = -12\cos 2t + 2\sin 2t$$

$$\phi'' + 4\phi = -12\cos 2t + 2\sin 2t + 4\left[3\cos 2t - \frac{1}{2}\sin 2t\right]$$

$$= -12\cos 2t + 2\sin 2t + 12\cos 2t - 2\sin 2t = 0$$

$$\phi(0) = 3\cos 0 = y(0)$$

$$\phi'(0) = -6\sin 0 - \cos 0 = -1 = y'(0)$$

Example

Find $\psi(t)$ when $g(t) = e^{-t}$.

$$G(s) = \frac{1}{s+1}$$

$$\begin{aligned}\Psi(s) = \frac{1}{(s^2+4)(s+1)} &= -\frac{1}{5}\frac{(s-1)}{(s^2+4)} + \frac{1}{5}\frac{1}{(s+1)} \\ &= -\frac{1}{5}\frac{s}{(s^2+4)} + \frac{1}{5}\frac{1}{(s^2+4)} + \frac{1}{5}\frac{1}{(s+1)} \\ &= -\frac{1}{5}\frac{s}{(s^2+4)} + \frac{1}{10}\frac{2}{(s^2+4)} + \frac{1}{5}\frac{1}{(s+1)}\end{aligned}$$

$$\psi(t) = -\frac{1}{5}\cos 2t + \frac{1}{10}\sin 2t + \frac{1}{5}e^{-t}$$

Example

$$\psi(t) = -\frac{1}{5}\cos 2t + \frac{1}{10}\sin 2t + \frac{1}{5}e^{-t}$$
$$\psi'(t) = \frac{2}{5}\sin 2t + \frac{1}{5}\cos 2t - \frac{1}{5}e^{-t}$$
$$\psi''(t) = \frac{4}{5}\cos 2t - \frac{2}{5}\sin 2t + \frac{1}{5}e^{-t}$$

$$\psi'' + 4\psi = \frac{4}{5}\cos 2t - \frac{2}{5}\sin 2t + \frac{1}{5}e^{-t} - \frac{4}{5}\cos 2t + 4\frac{1}{10}\sin 2t + \frac{4}{5}e^{-t}$$
$$= e^{-t}$$

$$\psi(0) = -\frac{1}{5} + \frac{1}{5} = 0$$
$$\psi'(0) = \frac{1}{5} - \frac{1}{5} = 0$$

Example

Find the impulse response $h(t)$:

$$H(s) = \frac{1}{s^2+4} = \frac{1}{2}\frac{2}{s^2+4}$$

$$h(t) = \frac{1}{2}\sin 2t$$
$$h'(t) = \cos 2t$$
$$h''(t) = -2\sin 2t$$

Example

For $t > 0$, $\delta(t) = 0$

$$h'' + 4h = -2\sin 2t + \frac{4}{2}\sin 2t = 0$$

Check at rest initial conditions:

$$h(0) = \frac{1}{2}\sin 0 = 0$$

$$h'(0) = \cos 0 = 1 \neq 0 \qquad \text{Not satisfied!}$$

This is due to a technicality of using $\delta(t)$ as the forcing function. It is a generalized function. The impulse response has the form

$$u_0(t)h(t) \qquad -\infty < t < \infty$$

where $u_0(t)$ is the Heaviside function or unit step at $t = 0$. $h'(0)$ is not always defined. The textbook ignores this detail.