

## Set 19: Laplace Transform and IVPs Part 4

Kyle A. Gallivan

Department of Mathematics

Florida State University

Ordinary Differential Equations

Fall 2009

1

### Convolution Integral

**Theorem 19.1.** Let  $f(t)$  and  $g(t)$  be two functions whose Laplace transforms are  $F(s)$  and  $G(s)$  respectively for  $s > a$ . If

$$H(s) = F(s)G(s), \quad s > a$$

then

$$h(t) = \mathcal{L}^{-1}\{H(s)\} = \int_0^t f(t-\tau)g(\tau)d\tau = \int_0^t f(\tau)g(t-\tau)d\tau = f * g$$

3

### Equivalence

Example of time domain and Laplace domain equivalent operations

Time Domain	Laplace Domain
linear combination	linear combination
differentiation	multiplication by $s$
multiplication by $t$	differentiation
translation by $c$	multiplication by $e^{-cs}$
multiplication by $e^{ct}$	translation by $c$

What about products of Laplace transforms?

2

### General Form

$$ay'' + by' + cy = g(t), y(0) = y_0, \quad y'(0) = y'_0$$

$\Downarrow$

$$Y(s) = \frac{(as + b)y_0 + ay'_0}{as^2 + bs + c} + \frac{G(s)}{as^2 + bs + c}$$

$$= \Phi(s) + \Psi(s) = \Phi(s) + H(s)G(s)$$

where  $G(s) = \mathcal{L}\{g\}$  and  $H(s) = (as^2 + bs + c)^{-1}$ .

4

### Another General Form

$$Y(s) = \Phi(s) + \Psi(s) = \Phi(s) + H(s)G(s)$$

$$y(t) = \phi(t) + \psi(t) = \phi(t) + h(t) * g(t)$$

- $\phi(t)$  is a solution to the homogeneous ODE that satisfies  $y(0) = y_0$ ,  $y'(0) = y'_0$
- $\psi(t)$  is a particular solution that satisfies the "at rest" conditions  $y(0) = y'(0) = 0$ .

5

### Another General Form

$$Y(s) = \Phi(s) + \Psi(s) = \Phi(s) + H(s)G(s)$$

$$y(t) = \phi(t) + \psi(t) = \phi(t) + h(t) * g(t)$$

- The transfer function  $H(s) = (as^2 + bs + c)^{-1}$  depends only on the ODE, i.e., on the system modeled.
- It gives the response to a forcing function specified by  $G(s)$  from an at rest condition.
- $G(s) = 1 \rightarrow g(t) = \delta(t)$  then  $Y(s) = H(s) \rightarrow y(t) = h(t) * \delta(t) = h(t)$  for  $t > 0$ .
- The transfer function  $H(s)$  is the Laplace transform of the impulse response  $h(t)$ .

6

### Example

$$y'' + 4y = g(t), \quad y(0) = 3, \quad y'(0) = -1$$

$$a = 1, \quad b = 0, \quad c = 4$$

$$\begin{aligned} Y(s) &= \frac{(as + b)y_0 + ay'_0}{as^2 + bs + c} + \frac{G(s)}{as^2 + bs + c} \\ &= \frac{3s - 1}{s^2 + 4} + \frac{G(s)}{s^2 + 4} \end{aligned}$$

$$\Phi(s) = H(s)(3s - 1), \quad \Psi(s) = H(s)G(s), \quad H(s) = \frac{1}{s^2 + 4}$$

7

### Example

Find  $\phi(t)$ :

$$\begin{aligned} \Phi(s) &= \frac{(3s - 1)}{s^2 + 4} \\ &= 3 \frac{s}{s^2 + 4} - \frac{1}{s^2 + 4} \\ &= 3 \frac{s}{s^2 + 4} - \frac{1}{2} \frac{1}{s^2 + 4} \\ \phi(t) &= 3 \cos 2t - \frac{1}{2} \sin 2t \end{aligned}$$

8

**Example**

$$\phi(t) = 3 \cos 2t - \frac{1}{2} \sin 2t$$

$$\phi'(t) = -6 \sin 2t - \cos 2t$$

$$\phi''(t) = -12 \cos 2t + 2 \sin 2t$$

$$\begin{aligned} \phi'' + 4\phi &= -12 \cos 2t + 2 \sin 2t + 4 \left[ 3 \cos 2t - \frac{1}{2} \sin 2t \right] \\ &= -12 \cos 2t + 2 \sin 2t + 12 \cos 2t - 2 \sin 2t = 0 \end{aligned}$$

$$\phi(0) = 3 \cos 0 = y(0)$$

$$\phi'(0) = -6 \sin 0 - \cos 0 = -1 = y'(0)$$

9

**Example**Find  $\psi(t)$  when  $g(t) = e^{-t}$ .

$$G(s) = \frac{1}{s+1}$$

$$\Psi(s) = \frac{1}{(s^2+4)(s+1)} = -\frac{1}{5} \frac{(s-1)}{(s^2+4)} + \frac{1}{5} \frac{1}{(s+1)}$$

$$= -\frac{1}{5} \frac{1}{(s^2+4)} + \frac{1}{5} \frac{1}{(s^2+4)} + \frac{1}{5} \frac{1}{(s+1)}$$

$$= -\frac{1}{5} \frac{1}{(s^2+4)} + \frac{1}{10} \frac{1}{(s^2+4)} + \frac{1}{5} \frac{1}{(s+1)}$$

$$\psi(t) = -\frac{1}{5} \cos 2t + \frac{1}{10} \sin 2t + \frac{1}{5} e^{-t}$$

10

**Example**

$$\psi(t) = -\frac{1}{5} \cos 2t + \frac{1}{10} \sin 2t + \frac{1}{5} e^{-t}$$

$$\psi'(t) = \frac{2}{5} \sin 2t + \frac{1}{5} \cos 2t - \frac{1}{5} e^{-t}$$

$$\psi''(t) = \frac{4}{5} \cos 2t - \frac{2}{5} \sin 2t + \frac{1}{5} e^{-t}$$

$$\begin{aligned} \psi'' + 4\psi &= \frac{4}{5} \cos 2t - \frac{2}{5} \sin 2t + \frac{1}{5} e^{-t} - \frac{4}{5} \cos 2t + 4 \frac{1}{10} \sin 2t + \frac{4}{5} e^{-t} \\ &= e^{-t} \end{aligned}$$

$$\psi(0) = -\frac{1}{5} + \frac{1}{5} = 0$$

$$\psi'(0) = \frac{1}{5} - \frac{1}{5} = 0$$

11

**Example**Find the impulse response  $h(t)$ :

$$H(s) = \frac{1}{s^2+4} = \frac{1}{2} \frac{1}{s^2+4}$$

$$h(t) = \frac{1}{2} \sin 2t$$

$$h'(t) = \cos 2t$$

$$h''(t) = -2 \sin 2t$$

12

### Example

For  $t > 0$ ,  $\delta(t) = 0$

$$h'' + 4h = -2 \sin 2t + \frac{4}{2} \sin 2t = 0$$

Check at rest initial conditions:

$$h(0) = \frac{1}{2} \sin 0 = 0$$

$$h'(0) = \cos 0 = 1 \neq 0 \quad \text{Not satisfied!}$$

This is due to a technicality of using  $\delta(t)$  as the forcing function. It is a generalized function. The impulse response has the form

$$u_0(t)h(t) \quad -\infty < t < \infty$$

where  $u_0(t)$  is the Heaviside function or unit step at  $t = 0$ .  $h'(0)$  is not always defined. The textbook ignores this detail.