Set 1: Basics

Kyle A. Gallivan

Department of Mathematics

Florida State University

Ordinary Differential Equations
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Differential Equations

Let

- Function $y : \mathbb{R} \to \mathbb{R}$, i.e., $t \mapsto y(t)$
- Derivatives

$$\frac{dy}{dt}(t), \quad \frac{d^2y}{dt^2}(t), \quad \frac{d^{(n)}y}{dt^{(n)}}(t)$$
$$y'(t), \quad y''(t), \ldots, y^{(n)}(t)$$

(Note: explicit dependence on t often dropped in notation)

- Differential equations relate y to one or more of its derivatives

$$F(t, y, y', y'', \ldots, y^{(n)}) = 0$$

Examples

- velocity of object in free fall $g > 0, \quad m > 0, \quad \gamma > 0$

$$m\frac{dv}{dt} = mg - \gamma v$$

- angle of pendulum, $L > 0$

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0$$

- RLC circuit capacitor charge for a given inductance (L), resistance (R), capacitance (C) and driving voltage $E(t)$.

$$LQ''(t) + RQ'(t) + \frac{1}{C}Q(t) = E(t)$$

Problem Characteristics

- An ordinary differential equation (ODE) involves one or more derivative with respect to a single independent variable.
- A partial differential equation (PDE) involves one or more partial derivative with respect to a multiple independent variables.

$$\frac{\partial u}{\partial t}(t, x) = \alpha^2 \frac{\partial^2 u}{\partial t^2}(t, x)$$

Problem Characteristics

- Systems of ordinary differential equations involve multiple dependent variables

$$y_1' = \omega y_2$$
$$y_2' = -\omega y_1$$

- The order of the differential equation is the order of the highest derivative

$$F(t, y, y', y'', \ldots, y^{(n)}) = 0$$

Problem Characteristics

- Linear differential equation

$$\alpha_0(t)y + \alpha_1(t)y' + \alpha_2(t)y'' + \cdots + \alpha_n(t)y^{(n)} = g(t)$$

- A nonlinear differential equation is any ODE not linear in form. For example,

$$y''' + 2e^t y'' + yy' = t^4$$

Problem Characteristics

- Explicit form is assumed:

$$F(t, y, y', y'', \ldots, y^{(n)}) = 0$$
$$\Downarrow$$
$$y^{(n)} = f(t, y, y', y'', \ldots, y^{(n-1)})$$

- Questions given an ODE:
  - Does a solution exist?
  - Is the solution unique?
  - Can we solve for the solution?
- "Solve" here means analytically/symbolically.
- Often a numerical solution is required.

A Simple Example

The ODE:

$$\frac{dy}{dt} = ay, \quad a \in \mathbb{R}, \quad y : \mathbb{R} \to \mathbb{R}$$

Try integrating :

$$\int \frac{dy}{dt}\, dt = \int a\, y\, dt = a \int y\, dt$$
$$y = a \int y\, dt$$

gives an integral equation not a solution.

A Simple Example

$$\frac{dy}{dt} = ay \quad \longrightarrow \quad \frac{1}{y}\frac{dy}{dt} = a$$

$$\frac{d}{dt}(\ln|y|) = a \quad \text{by chain rule}$$

$$\int \frac{d}{dt}(\ln|y|)dt = \int a\,dt \quad \longrightarrow \quad \ln|y| = at + c$$

$$|y| = e^{at+c} = e^c e^{at}$$

$$\therefore y = Ce^{at}$$

Easy to verify:

$$\frac{dy}{dt} = \frac{d}{dt}(Ce^{at}) = aCe^{at} = ay$$

A Simple Example

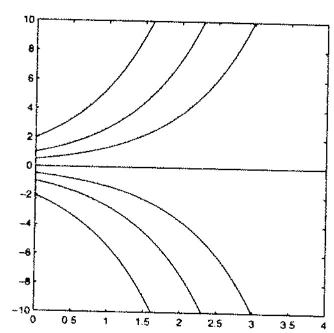
- Multiple solutions parameterized by arbitrary constant C.
- More information required to specify a single function as the solution.
- Initial value problem:

$$\frac{dy}{dt} = ay, \quad y(t_0) = y_0$$

- Solution

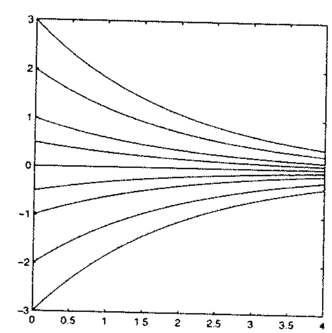
$$y = \frac{y_0}{e^{at_0}}e^{at}$$

Example: Integral Curves



$y' = y, y(0) = \pm 2,\ \pm 1,\ \pm 0.5$

Example: Integral Curves



$y' = -0.5y, y(0) = \pm 3\ , \pm 2,\ \pm 1,\ \pm 0.5$

Integral Curves

- Examining the intergral curves for a single ODE yields valuable information about the overall behavior of the different solutions.
- Behavior:
  - Equilibrium solutions
  - convergent solutions and regions
  - divergent solutions and regions
- direction fields
  - discussed in text Chapter 1
  - do not require solutions
  - plots slope as a function of t and y
  - gives a visual indication of behavior of solutions

Summary

- Initial Value Problem:

$$y^{(n)} = f(t, y, y', y'', \ldots, y^{(n-1)}), \quad y(t_0) = y_0$$

- y and f real valued functions
- Analytical/Symbolic solution
- Methods well-developed for linear problems
- Nonlinear problems are more difficult
- Methods depend on assumed form of the problem