

# **Set 2: First Order ODEs - Part 1**

**Kyle A. Gallivan**

**Department of Mathematics**

**Florida State University**

**Ordinary Differential Equations**

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## First Order ODE Initial Value Problem

- $t \in \mathbb{R} \mapsto y(t) \in \mathbb{R}$
- $(t, y) \in \mathbb{R}^2 \mapsto f(t, y) \in \mathbb{R}$
- $y'(t)$  is the highest order of derivative involved
- The initial value problem:

$$y' = f(t, y), \quad y(t_0) = y_0$$

- no general solution so we consider subclasses:
  - linear equations
  - separable equations
  - exact equations

## Undetermined Coefficients

The Undetermined Coefficients Method:

- hypothesizes a form of the solution that is a function of a set of coefficients
- substitutes the form of the solution into the differential equation
- solves for the unknown coefficients if possible

This will be used later on more complicated problems but we apply it to a simple problem here.

## Undetermined Coefficients

Consider the general first order linear constant-coefficient ODE

$$y' = ay - b \quad (1)$$

where  $a \neq 0$  and  $b$  are real constants.

- if  $b = 0$  we know  $y_1' = ay_1 \rightarrow y_1(t) = Ce^{at}$
- Equation (1) adds a constant to  $y_1'$  to get  $y'$
- Equation (1) says  $y'$  proportional to  $y$
- Hypothesize that there is a constant  $k$  such that

$$y(t) = y_1(t) + k$$

## Undetermined Coefficients

- substitute  $y(t) = y_1(t) + k$  into ODE RHS :

$$ay - b = ay_1(t) + ak - b = aCe^{at} + ak - b$$

- substitute  $y(t) = y_1(t) + k$  into ODE LHS :

$$\frac{dy}{dt} = \frac{d}{dt}(y_1(t) + k) = \frac{d}{dt}(Ce^{at}) + \frac{d}{dt}(k) = aCe^{at}$$

- equate RHS and LHS to get conditions on  $k$  and check for consistency with assumptions:

$$\text{if } aCe^{at} = aCe^{at} + ak - b \text{ then } 0 = ak - b$$

$$k = \frac{b}{a}$$

## Undetermined Coefficients

- Since  $a \neq 0$ ,  $k$  is a well-defined constant.
- Consistent with assumptions on solution, i.e., constant  $k$ .
- The function

$$y(t) = Ce^{at} + \frac{b}{a}$$

should be the general solution to the ODE.

- Verify:

$$y' = \frac{d}{dt} \left( Ce^{at} + \frac{b}{a} \right) = aCe^{at}$$

$$ay - b = a \left( Ce^{at} + \frac{b}{a} \right) - b = aCe^{at} + b - b = aCe^{at}$$

## Undetermined Coefficients

If the correct form of  $y(t)$  was not the hypothesis we should get a contradiction.

For example, hypothesize that there is a constant  $k$  such that

$$y(t) = y_1(t) + kt = Ce^{at} + kt$$

$$\text{LHS: } \frac{d}{dt}(Ce^{at} + kt) = aCe^{at} + k$$

$$\text{RHS: } ay - b = aCe^{at} + akt - b$$

$$\text{if LHS=RHS then } aCe^{at} + k = aCe^{at} + akt - b$$

$$b = k(at - 1) \rightarrow k = \frac{b}{at - 1}$$

This is a contradiction to the assumption that  $k$  is constant.

## Summary

First order linear constant coefficient ODE:

$$y' = ay - b$$

where  $a \in \mathbb{R}$ ,  $a \neq 0$  and  $b \in \mathbb{R}$

$$y(t) = Ce^{at} + \frac{b}{a}$$

general solution

$$y(t) = \left\{ e^{-at_0} \left( y_0 - \frac{b}{a} \right) \right\} e^{at} + \frac{b}{a}$$

IVP solution

for  $y(t_0) = y_0$

## Integrating Factor Method

Consider the extension to a slightly more complicated first order linear ODE:

$$y' + ay = g(t)$$

where  $t \in \mathbb{R} \mapsto g(t) \in \mathbb{R}$  is a continuous function and  $a \in \mathbb{R}$  is a constant.

An integrating factor  $\mu(t)$  is a function so that the solution of the ODE

$$\mu(t)y' + \mu(t)ay = \mu(t)g(t) \tag{2}$$

can be found “easily”.

## Basic Facts

Recall the product rule for derivatives:

$$\frac{d}{dt}(rs) = \frac{dr}{dt}s + r\frac{ds}{dt}$$

$$(rs)' = r's + rs'$$

This yields integration by parts:

$$\int rs' dt = rs - \int r's dt + c$$

## Integrating Factor

Compare LHS of the ODE containing  $\mu(t)$  and the product rule:

$$\text{LHS: } \mu(t)y' + \mu(t)ay$$

$$\text{PR: } \mu(t)y' + \mu'(t)y$$

They are equivalent if

$$\mu' = a\mu$$

## Integrating Factor

We know the solution for the ODE governing  $\mu(t)$ :

$$\mu' = a\mu \rightarrow \mu(t) = Ce^{at}$$

We can choose any  $C$  and for now we choose  $C = 1$

$$\therefore \mu(t) = e^{at}$$

is our integrating factor for ODE (2)

## Consequence of Use of Integrating Factor

$$\mu(t)y' + \mu(t)ay = \mu(t)g(t)$$

$$(e^{at})y' + (e^{at}a)y = e^{at}g(t)$$

$$(e^{at})y' + (e^{at})'y = (e^{at}y)' = e^{at}g(t)$$

$$\therefore e^{at}y = \int e^{at}g(t) dt + C$$

So if  $\int e^{at}g(t) dt$  can be found we have a solution to ODE (2).

$$y(t) = Ce^{-at} + e^{-at} \int e^{at}g(t) dt$$

## Consistency

If  $g(t) = -b$  we should have the solution derived earlier:

$$\begin{aligned}y(t) &= Ce^{-at} + e^{-at} \int e^{at} g(t) dt \\&= Ce^{-at} - be^{-at} \int e^{at} dt \\&= Ce^{-at} - be^{-at} \frac{e^{at}}{a} \\&= Ce^{-at} - \frac{b}{a}\end{aligned}$$

This is consistent with earlier solution of  $y' = -ay - b$ .

## Example

$$y' - 2y = 4 - t$$

$$\mu(t)y' - 2\mu(t)y = \mu(t)y' + \mu'(t)y = \mu(t)(4 - t)$$

$$\mu' = -2\mu \rightarrow \mu(t) = e^{-2t}$$

$$(e^{-2t}y)' = e^{-2t}(4 - t)$$

$$e^{-2t}y = C + \int e^{-2t}(4 - t) dt$$

$$e^{-2t}y = C + 4 \int e^{-2t} dt - \int te^{-2t} dt$$

## Example

$$e^{-2t}y = C + 4 \int e^{-2t} dt - \int te^{-2t} dt$$

$$4 \int e^{-2t} dt = -2 \int e^z dz = -2e^z = -2e^{-2t}$$

$$\therefore e^{-2t}y = C - 2e^{-2t} - \int te^{-2t} dt$$

## Example

$$\int r s' dt = r s - \int r' s dt + c$$

$$r = t \quad \text{and} \quad s' = e^{-2t}$$

$$r' = 1 \quad \text{and} \quad s = -\frac{1}{2}e^{-2t}$$

$$\begin{aligned} \int t e^{-2t} dt &= -\frac{1}{2} t e^{-2t} + \frac{1}{2} \int e^{-2t} dt \\ &= -\frac{1}{2} t e^{-2t} - \frac{1}{4} e^{-2t} + c \end{aligned}$$

## Example

$$\begin{aligned} e^{-2t}y &= C - 2e^{-2t} - \int te^{-2t} dt \\ &= C - 2e^{-2t} + \frac{1}{2}te^{-2t} + \frac{1}{4}e^{-2t} \end{aligned}$$

$$\therefore y(t) = Ce^{2t} - 2 + \frac{1}{2}t + \frac{1}{4} = Ce^{2t} + \frac{1}{2}t - \frac{7}{4}$$

## Example

Verify solution:

$$y(t) = Ce^{2t} + \frac{1}{2}t - \frac{7}{4}$$

$$\begin{aligned}y' - 2y &= \left(2Ce^{2t} + \frac{1}{2}\right) - 2\left(Ce^{2t} + \frac{1}{2}t - \frac{7}{4}\right) \\&= 2Ce^{2t} + \frac{1}{2} - 2Ce^{2t} - t + \frac{7}{2} \\&= 4 - t\end{aligned}$$

## Analysis of Behavior of Solution

Consider  $t \geq 0$ ,

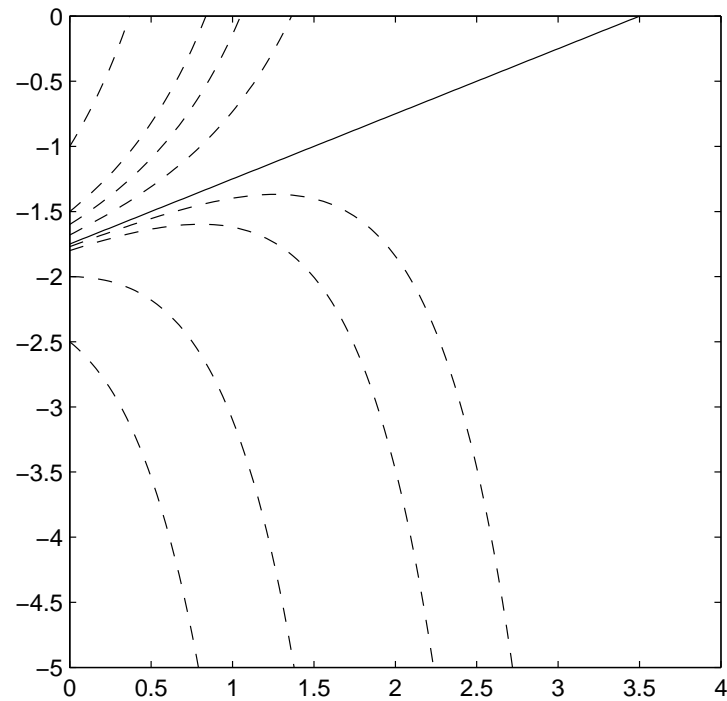
$$y(t) = Ce^{2t} + \frac{1}{2}t - \frac{7}{4}$$

$$y(0) = -\frac{7}{4} \rightarrow C = 0, \quad \therefore y(t) = \frac{1}{2}t - \frac{7}{4}$$

$$y(0) > -\frac{7}{4} \rightarrow C > 0, \quad \therefore y(t) \rightarrow \infty \text{ exponentially}$$

$$y(0) < -\frac{7}{4} \rightarrow C < 0, \quad \therefore y(t) \rightarrow -\infty \text{ exponentially}$$

## Example: Integral Curves



Integral curves for  $y' - 2y = 4 - t$

## Summary of Solutions

$$y' = b \longrightarrow y(t) = bt + C$$

$$y' = ay \longrightarrow y(t) = Ce^{at}$$

$$y' = ay - b \longrightarrow y(t) = Ce^{at} + \frac{b}{a}$$

$$y' + ay = g(t) \longrightarrow y(t) = Ce^{-at} + e^{-at} \int e^{at} g(t) dt$$

## General Linear Form

The general explicit first order linear ODE can be written

$$y' + p(t)y = g(t) \quad (3)$$

The integrating factor method derivation for  $p(t) = a$  is easily adapted to find a general solution:

if  $\mu' = p(t)\mu$  then

$$\mu(t)g(t) = \mu(t)y' + \mu(t)p(t)y = \mu(t)y' + \mu(t)'y = (\mu(t)y(t))'$$

$$\therefore \mu(t)y = C + \int \mu(t)g(t) dt$$

## General Linear Form

Hypothesize based on  $p(t) = a$  form

$$\mu(t) = \tilde{C}e^{z(t)}, \quad z(t) = \int p(t) dt$$

$$\mu'(t) = \tilde{C}e^{z(t)} z'(t) = \tilde{C}e^{z(t)} p(t) = \mu(t)p(t)$$

$$y(t) = \frac{C}{\mu(t)} + \frac{1}{\mu(t)} \int \mu(t)g(t) dt$$

Note two integrations needed to solve the problem.

## General Linear Form

For an IVP we have  $y(t_0) = y_0$ . Assume  $\tilde{C}$  is such that

$$\mu(t_0) = 1$$

We have

$$y(t) = \frac{C}{\mu(t)} + \frac{1}{\mu(t)} \int \mu(t)g(t) dt = \frac{C}{\mu(t)} + \frac{1}{\mu(t)} \int_{t_0}^t \mu(s)g(s) ds$$

$$y_0 = y(t_0) = \frac{C}{\mu(t_0)} + 0 \longrightarrow C = y_0$$

$$y(t) = \frac{1}{\mu(t)} \left\{ y_0 + \int_{t_0}^t \mu(s)g(s) ds \right\}$$

## Example

$$t \geq 1, \quad ty' + 2y = 4t^2 \rightarrow y' + \frac{2}{t}y = 4t$$

$$\int p(t) dt = 2 \int \frac{1}{t} dt = 2 \ln|t|$$

$$\mu(t) = \tilde{C}e^{2 \ln|t|} = \tilde{C}|t|^2 = \tilde{C}t^2$$

$$(t^2 y)' = t^2 y' + 2ty = 4t^3$$

$$t^2 y = t^4 + C \rightarrow y(t) = t^2 + \frac{C}{t^2}$$

$$y_1 = y(1) = 1 + C \rightarrow C = y_1 - 1$$

$$y(t) = t^2 + \frac{y_1 - 1}{t^2}$$

## Analysis of Behavior of Solution

We have

$$y(t) = t^2 + \frac{C}{t^2} = t^2 + \frac{y_1 - 1}{t^2}, \quad t \geq 1$$

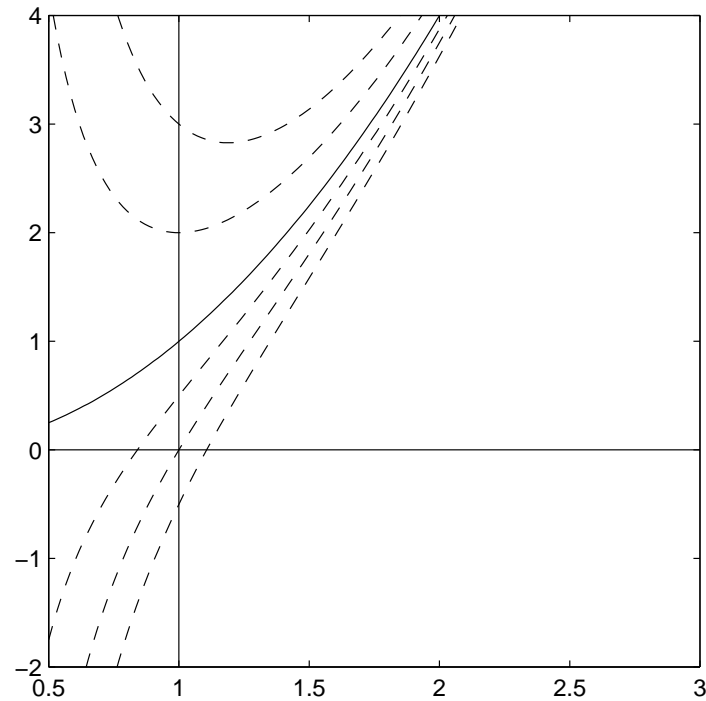
if  $y_1 = 1$  then  $C = 0$  and  $y(t) = t^2$

if  $y_1 > 1$  then  $C > 0$  and  $y(t) \rightarrow t^2$  from above

if  $y_1 < 1$  then  $C < 0$  and  $y(t) \rightarrow t^2$  from below

Note what happens as  $t_0 \rightarrow 0$ .

## Example: Integral Curves



Integral curves for  $y' + \frac{2}{t}y = 4t$  and  $t \geq 1$