Set 20: Systems of First Order Linear ODEs
Part 1

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Ordinary Differential Equations
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Systems of Linear ODEs

- $x_i : \mathbb{R} \rightarrow \mathbb{R}$, $i = 1, 2, 3$
- Suppose x_i' depends on x_i, $i = 1, 2, 3$ linearly

$$x_i'(t) = p_{i1}(t)x_1(t) + p_{i2}(t)x_2(t) + p_{i3}(t)x_3(t)$$

- System of 3 linear ODES defining 3 functions

$$\begin{aligned} x_1'(t) &= p_{11}(t)x_1(t) + p_{12}(t)x_2(t) + p_{13}(t)x_3(t) + g_1(t) \\ x_2'(t) &= p_{21}(t)x_1(t) + p_{22}(t)x_2(t) + p_{23}(t)x_3(t) + g_2(t) \\ x_3'(t) &= p_{31}(t)x_1(t) + p_{32}(t)x_2(t) + p_{33}(t)x_3(t) + g_3(t) \end{aligned}$$

Matrix Form

$$x'(t) = P(t)x(t) + g(t)$$

$$P(t) = \begin{pmatrix} p_{11}(t) & p_{12}(t) & p_{13}(t) \\ p_{21}(t) & p_{22}(t) & p_{23}(t) \\ p_{31}(t) & p_{32}(t) & p_{33}(t) \end{pmatrix}$$

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} \quad x'(t) = \begin{pmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{pmatrix} \quad g(t) = \begin{pmatrix} g_1(t) \\ g_2(t) \\ g_3(t) \end{pmatrix}$$

Matrix Vector Multiplication

$$x'(t) = P(t)x(t) + g(t)$$

$$P(t)x(t) = \begin{pmatrix} p_{11}(t) \\ p_{21}(t) \\ p_{31}(t) \end{pmatrix} x_1(t) + \begin{pmatrix} p_{12}(t) \\ p_{22}(t) \\ p_{32}(t) \end{pmatrix} x_2(t) + \begin{pmatrix} p_{13}(t) \\ p_{23}(t) \\ p_{33}(t) \end{pmatrix} x_3(t)$$

Linear combination of columns of the matrix with coefficients given by the corresponding elements of the vector.

Existence and Uniqueness

Theorem 20.1 (Textbook page 359)**.** *If* $p_{ij}(t)$, $1 \le i, \;\; j \le n$ *, and* $g_i(t)$, $1 \le i \le n$*, are continuous on an open interval* $\alpha < t < \beta$ *then there exists a unique solution* $x_1(t) = \phi_1(t), \ldots, x_n(t) = \phi_n(t)$ *to the initial value problem*

$$x'(t) = P(t)x(t) + g(t), \quad x(t_0) = x^{(0)}$$

for any $\alpha < t_0 < \beta$ *and* $x^{(0)} \in \mathbb{R}^n$*, that exists on the entire open interval.*

Higher Order Equations

- A single higher order linear equation can be transformed into a system of first order equations.
- $u''' + p(t)u'' + q(t)u' + r(t)u = f(t)$
- $x_1 = u$, $x_2 = x_1' = u'$, $x_3 = x_2' = u''$
- System of dimension 3

$$x_1'(t) = x_2(t)$$
$$x_2'(t) = x_3(t)$$
$$x_3'(t) = -r(t)x_1(t) - q(t)x_2(t) - p(t)x_3(t) + f(t)$$

Higher Order Equations

$$x_1'(t) = x_2(t)$$
$$x_2'(t) = x_3(t)$$
$$x_3'(t) = -r(t)x_1(t) - q(t)x_2(t) - p(t)x_3(t) + f(t)$$

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -r(t) & -q(t) & -p(t) \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ f(t) \end{pmatrix}$$

$$x'(t) = P(t)x(t) + g(t)$$

Systems of ODEs

Given the ODE $u^{(n)} = F(t, u,\ u', \dots, u^{(n-1)})$, the variable definitions $x_1 = u,\ \ x_2 = x_1' = u',\ \ x_3 = x_2' = u'', \dots x_n = x_{n-1}' = u^{(n-1)}$ convert the single nonlinear ODE of order n into a system of n nonlinear first order ODEs

$$
\begin{aligned}
x_1' &= x_2 \\
x_2' &= x_3 \\
&\vdots \\
x_{n-1}' &= x_n \\
x_n' &= F(t, x_1,\ x_2, \dots, x_{n-1})
\end{aligned}
$$

Systems of ODEs

The general form of a system of first order ODEs is

$$
\begin{aligned}
x_1' &= F_1(t, x_1,\ x_2, \ldots, x_{n-1}, x_n) \\
x_2' &= F_2(t, x_1,\ x_2, \ldots, x_{n-1}, x_n) \\
&\vdots \\
x_{n-1}' &= F_{n-1}(t, x_1,\ x_2, \ldots, x_{n-1}, x_n) \\
x_n' &= F_n(t, x_1,\ x_2, \ldots, x_{n-1}, x_n)
\end{aligned}
$$

Linear Constant Coefficient First Order System

Suppose $P(t)$ is a constant matrix A and for simplicity take $n = 2$.

$$x'(t) = P(t)x(t) + g(t) = Ax(t) + g(t)$$

$$A = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}$$

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \quad x'(t) = \begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} \quad g(t) = \begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix}$$

Homogeneous Problem

If $g(t) = 0\mathbb{R}^{\not\models}$ the problem is an homogeneous linear constant coefficient first order system.

$$x'(t) = Ax(t)$$

$$A = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}$$

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \qquad x'(t) = \begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix}$$

A Solution

Suppose we have a scalar r and a (constant(vector $v \in \mathbb{R}^2$ such that

$$Av = vr$$

i.e., the action of A on the vector v is equivalent to scaling by r.

r is an eigenvalue and v is an associated eigenvector of A.

A Solution

Consider:

$$x(t) = ve^{rt}$$

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} v_1 e^{rt} \\ v_2 e^{rt} \end{pmatrix}$$

$$x'(t) = \begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} v_1 r e^{rt} \\ v_2 r e^{rt} \end{pmatrix} = x(t)r$$

$$x' = x(t)r = vre^{rt} = Ave^{rt} = Ax$$

So an eigenvalue, eigenvector pair solves the homogeneous problem.

General Solution

- one solution

$$x(t) = ve^{rt}$$

- general solution to the homogeneous problem has a fundamental set of solutions so that all solutions to the homogeenous system can be written, for $n = 2$,

$$x(t) = x^{(1)}(t)c_1 + x^{(2)}(t)c_2$$

- What is the fundamental set of solutions for

$$x' = Ax$$

General Solution

- Want the general solution to the homogeneous system

$$x(t) = x^{(1)}(t)c_1 + x^{(2)}(t)c_2$$

- Need eigenvectors and eigenvalues
- Need linear independence
- We proceed as with scalar homogeneous problems but use vector-valued functions as solutions.

General Solution

- Suppose $x(t)$ is a solution to $x' = Ax$ on the open interval $\alpha < t < \beta$
- For any $\alpha < t_0 < \beta$ $x(t)$ is the unique solution with the value $x(t_0)$.
- Find unique constants so that $x(t) = x^{(1)}(t)c_1 + x^{(2)}(t)c_2$ for two solutions $x^{(1)}(t)$ and $x^{(2)}(t)$

General Solution

First note that given two solutions $x^{(1)}(t)$ and $x^{(2)}(t)$ and two constants c_1 and c_2 we have

$$x(t) = x^{(1)}(t)c_1 + x^{(2)}(t)c_2$$

$$x'(t) = \big[x^{(1)}(t)\big]'c_1 + \big[x^{(2)}(t)\big]'c_2$$

$$= Ax^{(1)}(t)c_1 + Ax^{(2)}(t)c_2 = A\big[x^{(1)}(t)c_1 + x^{(2)}(t)c_2\big]$$

$$= Ax(t)$$

So $x(t) = x^{(1)}(t)c_1 + x^{(2)}(t)c_2$ is also a solution to the homogeneous system.

General Solution

For any solution $x(t)$ to have this form we must have for any $\alpha < t_0 < \beta$ $x(t)$ a unique pair of constantsc_1 and c_2:

$$x(t_0) = x^{(1)}(t_0)c_1 + x^{(2)}(t_0)c_2$$

$$\begin{pmatrix} x_1(t_0) \\ x_2(t_0) \end{pmatrix} = \begin{pmatrix} x_1^{(1)}(t_0) & x_1^{(2)}(t_0) \\ x_2^{(1)}(t_0) & x_2^{(2)}(t_0) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$X(t_0)c = x(t_0)$$

Linear Algebra

Lemma 20.2. *An $n \times n$ linear system of equations $Av = b$ has a unique solution $v = A^{-1}b$ for any vector $b \in \mathbb{R}^n$ if any of the following equivalent conditions are true:*

- *All of the eigenvalues of A are nonzero.*
- $\det(A) \neq 0$
- *The linear system $Av = 0$ has a unique solution $v = 0$, i.e., the zero vector in $b \in \mathbb{R}^n$*
- *The columns of A are n linearly independent vectors.*

Note that if $Av = 0$ has a solution $v \neq 0$ then the columns of A are called linearly dependent.

General Solution

Definition 20.1. $x^{(1)}(t)$ and $x^{(2)}(t)$ are a fundamental set of solutions to the homogeneous system of ODEs $x' = Ax$ with $n = 2$ and $x(t) = x^{(1)}(t)c_1 + x^{(2)}(t)c_2$ is its general solution if

- At any point, t_0, in $\alpha < t < \beta$ the determinant $\det(X(t_0)) \neq 0$ where
$$X(t_0) = \begin{bmatrix} x^{(1)}(t_0) & x^{(2)}(t_0) \end{bmatrix} \in \mathbb{R}^{2\times 2}$$
- or equivalently the vectors $x^{(1)}(t_0)$ and $x^{(2)}(t_0)$ are linearly independent.

Note $\det(X(t)) = W[x^{(1)}(t),\ x^{(2)}(t)]$ is called the Wronskian of the set of vectors.

Fundamental Theorem

Theorem 20.3 (Textbook p. 387)**.** *If the vector functions* $x^{(1)}(t), \ldots, x^{(n)}(t)$ *are linearly independent vectors at each point in an open interval* $\alpha < t < \beta$ *and each solve*

$$x' = P(t)x$$

then any solution $x(t)$ *can be expressed as a unique linear combination of the* $x^{(i)}(t)$*. The vector functions* $x^{(1)}(t), \ldots, x^{(n)}(t)$ *are a fundamental set of solutions for the homogeneous ODE.*

Fundamental Theorem

Theorem 20.4 (Textbook p. 387)**.** *If the vector functions* $x^{(1)}(t), \ldots, x^{(n)}(t)$ *solve in an open interval* $\alpha < t < \beta$ *the ODE*

$$x' = P(t)x$$

then the Wronskian $W[x^{(1)}(t), \ldots, x^{(n)}(t)]$ *either is identically* 0 *or always nonzero on the entire interval.*

In other words, the solutions are linearly independent vectors at all points in the interval if they are linearly independent vectors at any single point.

Determinant Test $n = 2$

Definition 20.2. The determinant of the 2×2 matrix A is given by

$$\det(A) = \begin{vmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{vmatrix} = \alpha_{11}\alpha_{22} - \alpha_{21}\alpha_{12}$$

$$\det \begin{pmatrix} 1 & 1 \\ 2 & -3 \end{pmatrix} = \begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} = -3 - 2 = -5$$

$$\det \begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix} = \begin{vmatrix} 1 & -2 \\ 2 & -4 \end{vmatrix} = -4 + 4 = 0$$

Special Case for Two Vectors

- Note that two vectors, z_1 and z_2 are linearly depenendent if and only if one is a scalar multiple of the other, i.e., $z_1 = \alpha z_2$.
- For $n = 2$ this means the matrix with z_1 and z_2 as its columns has a determinant 0.

General Solution $n = 2$

Suppose we have two eigenvalue/eigenvector pairs, $(v^{(1)},\ r_1)$ and $(v^{(2)},\ r_2)$ we therefore have

$$x^{(1)}(t) = v^{(1)} e^{r_1 t} = \begin{pmatrix} v_1^{(1)} e^{r_1 t} \\ v_2^{(1)} e^{r_1 t} \end{pmatrix} \quad \text{and} \quad x^{(2)}(t) = v^{(2)} e^{r_2 t} = \begin{pmatrix} v_1^{(2)} e^{r_2 t} \\ v_2^{(2)} e^{r_2 t} \end{pmatrix}$$

$$\det(X(t)) = \begin{vmatrix} v_1^{(1)} e^{r_1 t} & v_1^{(2)} e^{r_2 t} \\ v_2^{(1)} e^{r_1 t} & v_2^{(2)} e^{r_2 t} \end{vmatrix} = v_1^{(1)} v_2^{(2)} e^{(r_1 + r_2)t} - v_1^{(2)} v_2^{(1)} e^{(r_1 + r_2)t}$$

$$= \left[v_1^{(1)} v_2^{(2)} - v_1^{(2)} v_2^{(1)} \right] e^{(r_1 + r_2)t} = \det(X(0)) e^{(r_1 + r_2)t}$$

So $\det(X(t)) \neq 0 \leftrightarrow \det(X(0)) \neq 0$ or equivalently, $v^{(1)}$ and $v^{(2)}$ are linearly independent.

Independence

Theorem 20.5. *If* $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$ *are eigenvalues of* $A \in \mathbb{R}^{n\times n}$ *with eigenvectors* $v^{(1)},\ v^{(2)}, \ldots, v^{(n)}$*, respectively, then*

- *the eigenvectors are linearly independent;*
- *the matrix* $V \in \mathbb{R}^{n\times n}$ *whose* i*-th column is* $v^{(i)}$ *is nonsingular, i.e.,* V^{-1} *exists uniquely;*
- *The system of linear equations* $Vc = b$ *has a unique solution* $c = V^{-1}b$ *for any* $b \in \mathbb{R}^{n\times n}$*.*

Fundamental Theorem

Theorem 20.6. *The homogeneous linear constant coefficient system of first order ODEs*

$$x' = Ax$$

has a fundamental set of solutions

$$x^{(i)}(t) = v^{(i)} e^{r_i t}, \quad 1 \le i \le n$$

and general solution

$$x(t) = x^{(1)}(t)c_1 + \cdots + x^{(n)}(t)c_n$$

if $(v^{(1)},\ r_1), \ldots, (v^{(n)},\ r_n)$ *are eigenvector/eigenvalue pairs with* $r_1 \ne r_2 \ne \cdots \ne r_n$.

Independent or Dependent

Tests for linear independence or dependence of a set of vectors $x^{(i)} \in \mathbb{R}^n$ for $i = 1, \ldots, n$

- Solve $Xc = 0$ for c.
  - If there is a nonzero solution then the vectors are dependent. In this case there will be more than one $c \neq 0$.
  - If $c = 0$ is the only solution then the vectors are independent.
- Compute the determinant of X, $\det(X)$.
  - If $\det(X) = 0$ then the vectors are dependent.
  - If $\det(X) \neq 0$ then the vectors are independent.
- Check the eigenvalues of X.
  - If at least one eigenvalue is 0 then the vectors are dependent.
  - If all eigenvalues are nonzero then the vectors are independent.

Determinant Test

Definition 20.3. The determinant of the 2×2 matrix A is given by

$$\det(A) = \begin{vmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{vmatrix} = \alpha_{11}\alpha_{22} - \alpha_{21}\alpha_{12}$$

Example

$$\det \begin{pmatrix} 1 & 1 \\ 2 & -3 \end{pmatrix} = \begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} = -3 - 2 = -5$$

$$\det \begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix} = \begin{vmatrix} 1 & -2 \\ 2 & -4 \end{vmatrix} = -4 + 4 = 0$$

Note it is also easy to see that one column is or is not a scalar mulitple of the other.

Eigenvalues and Eigenvector

Definition 20.4. If A is an $n \times n$ matrix then the nonzero vector x and scalar λ are an eigenvector/eigenvector pair if

$$Ax = x\lambda$$

We have

$$Ax = x\lambda \rightarrow (A - \lambda I)x = 0$$

$$\therefore \det(A - \lambda I) = 0$$

Eigenvalues and Eigenvector

- $\det(A - \lambda I)$ is a polynomial of degree n in the variable λ
- The eigenvalues are the roots of the polynomial.
- Any nonzero vector that solves, the system of equations

$$(A - \lambda I)x = 0$$

  is an eigenvector associated with λ.
- If x is an eigenvector associated with eigenvalue λ then so is αx for any scalar α.
- If x_1 and $x_2 \neq \alpha x_1$ are eigenvectors associated with eigenvalue λ then so is $\alpha_1 x_1 + \alpha_2 x_2$.

Example

Example 4 Textbook page 379

$$A = \begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix}$$

$$\det \begin{pmatrix} 3-\lambda & -1 \\ 4 & -2-\lambda \end{pmatrix} = -(3-\lambda)(2+\lambda)+4$$

$$= \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1) = 0$$

$$\therefore \lambda_1 = 2 \quad \text{and} \quad \lambda_2 = -1$$

Example

Example 4 Textbook page 379

$$A - 2I = \begin{pmatrix} 1 & -1 \\ 4 & -4 \end{pmatrix}$$

add -4 times row 1 to row 2

$$\begin{pmatrix} 1 & -1 & | & 0 \\ 4 & -4 & | & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & | & 0 \\ 0 & 0 & | & 0 \end{pmatrix} \to x_1 = x_2$$

So any vector with $x_1 = x_2$ with x_2 an arbitrary value is an eigenvector for $\lambda_1 = 2$.

Example

Example 4 Textbook page 379

$$A + I = \begin{pmatrix} 4 & -1 \\ 4 & -1 \end{pmatrix}$$

add row 1 to row 2

$$\begin{pmatrix} 4 & -1 & | & 0 \\ 4 & -1 & | & 0 \end{pmatrix} \to \begin{pmatrix} 4 & -1 & | & 0 \\ 0 & 0 & | & 0 \end{pmatrix} \to 4x_1 = x_2$$

So any vector with $x_1 = 0.25x_2$ with x_2 an arbitrary value is an eigenvector for $\lambda_1 = -1$.

Independence

Note that

$$\lambda_1 = 2, \quad x^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \lambda_2 = -1, \quad x^{(2)} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

$$x^{(1)} \quad \text{and} \quad x^{(2)} \quad \text{are linearly independent}$$

This is consistent with Theorem 20.5.

Initial Value Problem for the Example

Suppose we impose the initial conditions $x_1(0) = 1$ and $x_2(0) = 1$

$$x(t) = x^{(1)}(t)c_1 + x^{(2)}(t)c_2 = \begin{pmatrix} e^{2t} & e^{-t} \\ e^{2t} & 4e^{-t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$c_1 = 1, \quad c_2 = 0$$

$$x(t) = x^{(1)}(t) = \begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix} \quad \text{solves the IVP}$$

Problems with $n \geq 3$

- For this class exams will only cover ODE systems with $n = 2$.
- The textbook has discussions and examples of determining eigenvalues/eigenvectors and solving linear systems with $n = 3$ and $n = 4$. Read Sections 7.2 and 7.3.
- The rest of this set gives some examples.
- The homework will include other problems with $n = 3$.

Determinant Test

Definition 20.5. The determinant, $\det(A)$, of a 3×3 matrix A is given by:

$$\det(A) = \begin{vmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{vmatrix}$$

$$= (-1)^{i+1} \det(A_{i1})\alpha_{i1} + (-1)^{i+2} \det(A_{i2})\alpha_{i2} + (-1)^{i+3} \det(A_{i3})\alpha_{i3}$$

$$= (-1)^{1+j} \det(A_{1j})\alpha_{1j} + (-1)^{2+j} \det(A_{2j})\alpha_{2j} + (-1)^{3+j} \det(A_{3j})\alpha_{3j}$$

where A_{ij} is the 2×2 matrix resulting from removing row i and column j from A.

Example

Use row 1:

$$\det(A) = \begin{vmatrix} 1 & -2 & 3 \\ -1 & 2 & -2 \\ 2 & -1 & -1 \end{vmatrix}$$

$$= (-1)^{1+1}(1)\begin{vmatrix} 2 & -2 \\ -1 & -1 \end{vmatrix} + (-1)^{1+2}(-2)\begin{vmatrix} -1 & -2 \\ 2 & -1 \end{vmatrix} + (-1)^{1+3}(3)\begin{vmatrix} -1 & 2 \\ 2 & -1 \end{vmatrix}$$

$$= 0 + 10 - 9 = 1 \neq 0$$

Example

Use column 1:

$$\det(A) = \begin{vmatrix} 1 & -2 & 3 \\ -1 & 2 & -2 \\ 2 & -1 & -1 \end{vmatrix}$$

$$= (-1)^{1+1}(1)\begin{vmatrix} 2 & -2 \\ -1 & -1 \end{vmatrix} + (-1)^{2+1}(-1)\begin{vmatrix} -2 & 3 \\ -1 & -1 \end{vmatrix} + (-1)^{3+1}(2)\begin{vmatrix} -2 & 3 \\ 2 & -2 \end{vmatrix}$$

$$= 0 + 5 - 4 = 1 \neq 0$$

Example

Use row 1:

$$\det(A) = \begin{vmatrix} 1 & -2 & 3 \\ -1 & 1 & -2 \\ 2 & -1 & 3 \end{vmatrix}$$

$$= (-1)^{1+1}(1)\begin{vmatrix} 1 & -2 \\ -1 & 3 \end{vmatrix} + (-1)^{1+2}(-2)\begin{vmatrix} -1 & -2 \\ 2 & 3 \end{vmatrix} + (-1)^{1+3}(3)\begin{vmatrix} -1 & 1 \\ 2 & -1 \end{vmatrix}$$

$$= (3-2) + 2(-3+4) + 3(1-2)$$
$$= 1 + 2 - 3 = 0$$

Example

Find the eigenvalues.

$$A = \begin{pmatrix} 3 & 2 & 2 \\ 1 & 4 & 1 \\ -2 & -4 & -1 \end{pmatrix} \quad \det \begin{pmatrix} 3-\lambda & 2 & 2 \\ 1 & 4-\lambda & 1 \\ -2 & -4 & -1-\lambda \end{pmatrix}$$

Example

$$\det(A - \lambda I) = (-1)^{1+1}(3-\lambda)\begin{vmatrix} 4-\lambda & 1 \\ -4 & -1-\lambda \end{vmatrix}$$

$$+(-1)^{1+2}(2)\begin{vmatrix} 1 & 1 \\ -2 & -1-\lambda \end{vmatrix} + (-1)^{1+3}(2)\begin{vmatrix} 1 & 4-\lambda \\ -2 & -4 \end{vmatrix}$$

$$= (3-\lambda)(\lambda^2 - 3\lambda) + 2(\lambda - 1) + 2(-2\lambda + 4)$$

$$= -\lambda^3 + 6\lambda^2 - 11\lambda + 6$$

$$\lambda_1 = 1, \quad \lambda_2 = 2, \quad \lambda_3 = 3$$

Example

Let $n = 3$ and consider

$$x^{(1)} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \quad x^{(2)} = \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}, \quad x^{(3)} = \begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix}$$

Solve the system

$$\begin{pmatrix} 1 & -2 & 3 \\ -1 & 1 & -2 \\ 2 & -1 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Example

Use row combinations to transform system of equations (this also works for nonzero righthand side vectors):

add row 1 to row 2; add -2 times row 1 to row 3

$$\begin{pmatrix} 1 & -2 & 3 & | & 0 \\ -1 & 1 & -2 & | & 0 \\ 2 & -1 & -1 & | & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 3 & | & 0 \\ 0 & -1 & 1 & | & 0 \\ 0 & 3 & -7 & | & 0 \end{pmatrix}$$

multiply row 2 by -1

$$\begin{pmatrix} 1 & -2 & 3 & | & 0 \\ 0 & -1 & 1 & | & 0 \\ 0 & 3 & -7 & | & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 3 & | & 0 \\ 0 & 1 & -1 & | & 0 \\ 0 & 3 & -7 & | & 0 \end{pmatrix}$$

Example

add 3 times row 2 to row 3

$$\begin{pmatrix} 1 & -2 & 3 & | & 0 \\ 0 & 1 & -1 & | & 0 \\ 0 & 3 & -7 & | & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 3 & | & 0 \\ 0 & 1 & -1 & | & 0 \\ 0 & 0 & -4 & | & 0 \end{pmatrix}$$

divide row 3 by -4

$$\begin{pmatrix} 1 & -2 & 3 & | & 0 \\ 0 & 1 & -1 & | & 0 \\ 0 & 0 & -4 & | & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 3 & | & 0 \\ 0 & 1 & -1 & | & 0 \\ 0 & 0 & 1 & | & 0 \end{pmatrix}$$

Example

$$\begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$c_1 - 2c_2 + 3c_3 = 0$$

$$c_2 - c_3 = 0$$

$$c_3 = 0$$

$c_1 = c_2 = c_3 = 0$ is unique solution so the vectors are independent.

Example

Let $n = 3$ and consider

$$x^{(1)} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \quad x^{(2)} = \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}, \quad x^{(3)} = \begin{pmatrix} 3 \\ -2 \\ 3 \end{pmatrix}$$

Solve the system

$$\begin{pmatrix} 1 & -2 & 3 \\ -1 & 1 & -2 \\ 2 & -1 & 3 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Example

due to the position modified in the matrix the same row combinations transform system of equations:

add row 1 to row 2; add -2 times row 1 to row 3

$$\begin{pmatrix} 1 & -2 & 3 & | & 0 \\ -1 & 1 & -2 & | & 0 \\ 2 & -1 & 3 & | & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -2 & 3 & | & 0 \\ 0 & -1 & 1 & | & 0 \\ 0 & 3 & -3 & | & 0 \end{pmatrix}$$

multiply row 2 by -1

$$\begin{pmatrix} 1 & -2 & 3 & | & 0 \\ 0 & -1 & 1 & | & 0 \\ 0 & 3 & -7 & | & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -2 & 3 & | & 0 \\ 0 & 1 & -1 & | & 0 \\ 0 & 3 & -3 & | & 0 \end{pmatrix}$$

Example

add 3 times row 2 to row 3

$$\begin{pmatrix} 1 & -2 & 3 & | & 0 \\ 0 & 1 & -1 & | & 0 \\ 0 & 3 & -3 & | & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -2 & 3 & | & 0 \\ 0 & 1 & -1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

Example

$$\begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$c_1 - 2c_2 + 3c_3 = 0$$
$$c_2 - c_3 = 0$$
$$0 = 0$$

c_3 is arbitrary and then c_1 and c_2 follow.

So $c_1 = -1$, $c_2 = 1$, and $c_3 = 1$ is a nonzero solution. The vectors are dependent.

Example

$$x^{(1)} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \quad x^{(2)} = \begin{pmatrix} -2 \\ 2 \\ -1 \end{pmatrix}, \quad x^{(3)} = \begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -2 & 3 \\ -1 & 2 & -2 \\ 2 & -1 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Example

add row 1 to row 2; add -2 times row 1 to row 3

$$\begin{pmatrix} 1 & -2 & 3 & | & 0 \\ -1 & 2 & -2 & | & 0 \\ 2 & -1 & -1 & | & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 3 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 3 & -7 & | & 0 \end{pmatrix}$$

swap rows 2 and 3

$$\begin{pmatrix} 1 & -2 & 3 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 3 & -7 & | & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 3 & | & 0 \\ 0 & 3 & -7 & | & 0 \\ 0 & 0 & 1 & | & 0 \end{pmatrix}$$

Example

Divide row 2 by 3

$$\begin{pmatrix} 1 & -2 & 3 & | & 0 \\ 0 & 3 & -7 & | & 0 \\ 0 & 0 & 1 & | & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 3 & | & 0 \\ 0 & 1 & -7/3 & | & 0 \\ 0 & 0 & 1 & | & 0 \end{pmatrix}$$

$$c_1 - 2c_2 + 3c_3$$

$$c_2 - \frac{7}{3}c_3 = 0$$

$$c_3 = 0$$

So $c_1 = c_2 = c_3 = 0$ is the unique solution.