Set 21: Systems of First Order Linear Equations Part 2

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Ordinary Differential Equations
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Complex Eigenvalues

- If a real matrix A has a complex eigenvalue $\lambda = \alpha + i\beta$ then $\overline{\lambda} = \alpha - i\beta$ must also be an eigenvalue.
- This follows from the fact that $\det(A - \lambda I)$ is a polynomial of degree n with real coefficients.
- We want real valued solutions to

$$x' = Ax$$

- The eigenvectors of A corresponding to λ and $\overline{\lambda}$ will be complex.

Eigenvectors of Conjugate Eigenvalues

Lemma 21.1. *If the matrix* $A \in \mathbb{R}^{n \times n}$ *has conjugate eigenvalues* $\lambda_1 = \alpha + i\beta$ *and* $\lambda_2 = \overline{\lambda} = \alpha - i\beta$ *with eigenvectors* $z^{(1)}$ *and* $z^{(2)}$ *respectively then*

$$z^{(1)} = a + ib \quad \textit{and} \quad z^{(2)} = a - ib = \overline{z^{(1)}}$$

where $a \in \mathbb{R}^n$ *and* $b \in \mathbb{R}^n$

Eigenvectors of Conjugate Eigenvalues

Proof. This follows from direct computation. Since for any two complex scalars we know $\overline{(\gamma\delta)} = \overline{(\gamma} \ \overline{\delta)}$ we have

$$Av^{(1)} = z^{(1)}\lambda_1$$

$$\overline{Az^{(1)}} = \overline{z^{(1)}\lambda_1}$$

$$\overline{A} \ \overline{z^{(1)}} = \overline{z^{(1)}} \ \overline{\lambda_1}$$

$$A\overline{z^{(1)}} = \overline{z^{(1)}} \ \overline{\lambda_1}$$

$$Az^{(2)} = z^{(2)}\lambda_2$$

□

Solutions

$$x^{(1)}(t) = z^{(1)} e^{(\alpha + i\beta)t} = (a + ib) e^{(\alpha + i\beta)t}$$

$$= e^{\alpha t}(a \cos \beta t - b \sin \beta t) + i e^{\alpha t}(a \sin \beta t + b \cos \beta t) = u(t) + i v(t)$$

If can be shown that $u(t)$ and $v(t)$ are linearly independent solutions to the ODE.

Solutions

If the matrix $A \in \mathbb{R}^{n\times n}$ has conjugate eigenvalues $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \overline{\lambda} = \alpha - i\beta$ with eigenvectors $z^{(1)} = a + ib$ and $z^{(2)} = a - ib$ and distinct eigenvalues $\lambda_3, \ldots, \lambda_n$ with real eigenvectors $v^{(3)}, \ldots v^{(n)}$ then a general solution of $x' = Ax$ is

$$c_1 u(t) + c_2 v(t) + c_2 v^{(3)} e^{\lambda_3 t} + \cdots + c_2 v^{(n)} e^{\lambda_n t}$$
$$u(t) = e^{\alpha t}(a \cos \beta t - b \sin \beta t)$$
$$v(t) = e^{\alpha t}(a \sin \beta t + b \cos \beta t)$$

The result is straightforward to adapt to multiple conjugate pairs and distinct real eigenvalues.

Example

Textbook p. 401

$$x' = Ax$$

$$A = \begin{pmatrix} -0.5 & 1 \\ -1 & -0.5 \end{pmatrix}$$

$$\det(A - \lambda I) = \begin{pmatrix} -0.5 - \lambda & 1 \\ -1 & -0.5 - \lambda \end{pmatrix} = \lambda^2 + \lambda + \frac{5}{4}$$

$$\lambda_{\pm} = -0.5 \pm 0.5i\sqrt{4} = -0.5 \pm i$$

Example

$$(A - \lambda_1 I)z^{(1)} = 0$$

$$\begin{pmatrix} -0.5 - (-0.5 + i) & 1 \\ -1 & -0.5 - (-0.5 + i) \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore z_2 = iz_1$$

$$z^{(1)} = \begin{pmatrix} 1 \\ i \end{pmatrix} \rightarrow x^{(1)}(t) = \begin{pmatrix} 1 \\ i \end{pmatrix} e^{(-0.5+i)t}$$

Example

$$x^{(1)}(t) = \begin{pmatrix} 1 \\ i \end{pmatrix} e^{(-0.5+i)t} = \begin{pmatrix} 1 \\ i \end{pmatrix} e^{-0.5t}(\cos t + i \sin t)$$

$$x^{(1)}(t) = \begin{pmatrix} e^{-0.5t} \cos t \\ -e^{-0.5t} \sin t \end{pmatrix} + i \begin{pmatrix} e^{-0.5t} \sin t \\ e^{-0.5t} \cos t \end{pmatrix}$$

$$u(t) = \begin{pmatrix} e^{-0.5t} \cos t \\ -e^{-0.5t} \sin t \end{pmatrix} \qquad v(t) = \begin{pmatrix} e^{-0.5t} \sin t \\ e^{-0.5t} \cos t \end{pmatrix}$$

Example

$$W[u,v](t) = \begin{vmatrix} e^{-0.5t}\cos t & e^{-0.5t}\sin t \\ -e^{-0.5t}\sin t & e^{-0.5t}\cos t \end{vmatrix}$$

$$= e^{-t}(\cos^2 t + \sin^2 t) \neq 0$$

$$x(t) = c_1 \begin{pmatrix} e^{-0.5t}\cos t \\ -e^{-0.5t}\sin t \end{pmatrix} + c_2 \begin{pmatrix} e^{-0.5t}\sin t \\ e^{-0.5t}\cos t \end{pmatrix}$$