Set 21: Systems of First Order Linear Equations Part 2

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Ordinary Differential Equations
Fall 2009

Complex Eigenvalues

- If a real matrix A has a complex eigenvalue $\lambda = \alpha + i\beta$ then $\overline{\lambda} = \alpha - i\beta$ must also be an eigenvalue.
- This follows from the fact that $\det(A - \lambda I)$ is a polynomial of degree n with real coefficients.
- We want real valued solutions to

$$x' = Ax$$

- The eigenvectors of A corresponding to λ and $\overline{\lambda}$ will be complex.

Eigenvectors of Conjugate Eigenvalues

Lemma 21.1. *If the matrix $A \in \mathbb{R}^{n \times n}$ has conjugate eigenvalues $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \overline{\lambda} = \alpha - i\beta$ with eigenvectors $z^{(1)}$ and $z^{(2)}$ respectively then*

$$z^{(1)} = a + ib \quad \textit{and} \quad z^{(2)} = a - ib = \overline{z^{(1)}}$$

where $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^n$

Eigenvectors of Conjugate Eigenvalues

Proof. This follows from direct computation. Since for any two complex scalars we know $\overline{(\gamma\delta)} = \overline{(\gamma} \ \overline{\delta)}$ we have

$$Av^{(1)} = z^{(1)}\lambda_1$$
$$\overline{Az^{(1)}} = \overline{z^{(1)}\lambda_1}$$
$$\overline{A} \ \overline{z^{(1)}} = \overline{z^{(1)}} \ \overline{\lambda_1}$$
$$A\overline{z^{(1)}} = \overline{z^{(1)}} \ \overline{\lambda_1}$$
$$Az^{(2)} = z^{(2)}\lambda_2$$

□

Solutions

$$x^{(1)}(t) = z^{(1)}e^{(\alpha+i\beta)t} = (a+ib)e^{(\alpha+i\beta)t}$$
$$= e^{\alpha t}(a\cos\beta t - b\sin\beta t) + ie^{\alpha t}(a\sin\beta t + b\cos\beta t) = u(t) + iv(t)$$

If can be shown that $u(t)$ and $v(t)$ are linearly independent solutions to the ODE.

Solutions

If the matrix $A \in \mathbb{R}^{n\times n}$ has conjugate eigenvalues $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \overline{\lambda} = \alpha - i\beta$ with eigenvectors $z^{(1)} = a + ib$ and $z^{(2)} = a - ib$ and distinct eigenvalues $\lambda_3, \ldots, \lambda_n$ with real eigenvectors $v^{(3)}, \ldots v^{(n)}$ then a general solution of $x' = Ax$ is

$$c_1u(t) + c_2v(t) + c_2v^{(3)}e^{\lambda_3 t} + \cdots + c_2v^{(n)}e^{\lambda_n t}$$
$$u(t) = e^{\alpha t}(a\cos\beta t - b\sin\beta t)$$
$$v(t) = e^{\alpha t}(a\sin\beta t + b\cos\beta t)$$

The result is straightforward to adapt to multiple conjugate pairs and distinct real eigenvalues.

Example

Textbook p. 401

$$x' = Ax$$
$$A = \begin{pmatrix} -0.5 & 1 \\ -1 & -0.5 \end{pmatrix}$$
$$\det(A - \lambda I) = \begin{pmatrix} -0.5-\lambda & 1 \\ -1 & -0.5-\lambda \end{pmatrix} = \lambda^2 + \lambda + \frac{5}{4}$$
$$\lambda_\pm = -0.5 \pm 0.5\sqrt{4} = -0.5 \pm i$$

Example

$$(A - \lambda_1 I)z^{(1)} = 0$$
$$\begin{pmatrix} -0.5-(-0.5+i) & 1 \\ -1 & -0.5-(-0.5+i) \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\therefore z_2 = iz_1$$
$$z^{(1)} = \begin{pmatrix} 1 \\ i \end{pmatrix} \to x^{(1)}(t) = \begin{pmatrix} 1 \\ i \end{pmatrix} e^{(-0.5+i)t}$$

Example

$$x^{(1)}(t) = \begin{pmatrix} 1 \\ i \end{pmatrix} e^{(-0.5+i)t} = \begin{pmatrix} 1 \\ i \end{pmatrix} e^{-0.5t}(\cos t + i \sin t)$$

$$x^{(1)}(t) = \begin{pmatrix} e^{-0.5t} \cos t \\ -e^{-0.5t} \sin t \end{pmatrix} + i \begin{pmatrix} e^{-0.5t} \sin t \\ e^{-0.5t} \cos t \end{pmatrix}$$

$$u(t) = \begin{pmatrix} e^{-0.5t} \cos t \\ -e^{-0.5t} \sin t \end{pmatrix} \quad v(t) = \begin{pmatrix} e^{-0.5t} \sin t \\ e^{-0.5t} \cos t \end{pmatrix}$$

Example

$$W[u, v](t) = \begin{vmatrix} e^{-0.5t} \cos t & e^{-0.5t} \sin t \\ -e^{-0.5t} \sin t & e^{-0.5t} \cos t \end{vmatrix}$$

$$= e^{-t}(\cos^2 t + \sin^2 t) \neq 0$$

$$x(t) = c_1 \begin{pmatrix} e^{-0.5t} \cos t \\ -e^{-0.5t} \sin t \end{pmatrix} + c_2 \begin{pmatrix} e^{-0.5t} \sin t \\ e^{-0.5t} \cos t \end{pmatrix}$$