Set 22: Systems of First Order Linear Equations Part 3

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Ordinary Differential Equations
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Repeated Eigenvalues

- Since $\det(A - \lambda I)$ is a polynomial it is possible to have a root with a multiplicity greater than 1, i.e., the polynomial has a factor

$$(\lambda - \lambda_1)^m, \quad m > 1$$

- m is the algebraic multiplicity of the eigenvalue λ_1
- λ_1 can have at most m linearly independent eigenvectors
- λ_1 can have as few as 1 eigenvector
- the number of linearly independent eigenvectors, g, for λ_1 is its geometric multiplicity
- When $g < m$ we must generate other linearly independent vectors to associate with λ_1 and form a piece of the general solution to the homogeneous ODE

Example

$$A_1 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

Repeated Eigenvalues

Suppose λ is an eigenvalue of $A \in \mathbb{R}^{n \times n}$ with $m = 2$ and $g = 1$

$$Av = v\lambda \rightarrow x^{(1)}(t) = ve^{\lambda t}$$

$$\text{hypothesis:} \quad x^{(2)}(t) = vte^{\lambda t} + pe^{\lambda t}$$

$$\left(x^{(2)}\right)'(t) = ve^{\lambda t} + p\lambda e^{\lambda t} + v\lambda te^{\lambda t}$$

$$Ax^{(2)}(t) = Avte^{\lambda t} + Ape^{\lambda t} = v\lambda te^{\lambda t} + Ape^{\lambda t}$$

$$(v + p\lambda)e^{\lambda t} = Ape^{\lambda t}$$

$$\therefore v = Ap - p\lambda \rightarrow (A - \lambda I)p = v$$

Repeated Eigenvalues

- must solve $(A - \lambda I)p = v$
- $\det(A - \lambda I) = 0$ so in general no guarantee of a solution
- since v is an eigenvector a solution p exists
- p and v are linearly independent
- $\lambda_1 = \lambda_2 = \lambda$ and, e.g., $\lambda_3 \neq \cdots \neq \lambda_n \neq \lambda$

$$x(t) = c_1 x^{(1)}(t) + c_2 x^{(2)}(t) + \sum_{i=3}^{n} c_i x^{(i)}(t)$$

$$= c_1 v e^{\lambda t} + c_2 \left(vte^{\lambda t} + pe^{\lambda t}\right) + \sum_{i=3}^{n} c_i v^{(i)} e^{\lambda_i t}$$

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Example

$$A = \begin{pmatrix} -3 & 1 \\ 0 & -3 \end{pmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} -3-\lambda & 1 \\ 0 & -3-\lambda \end{vmatrix} = (-3-\lambda)(-3-\lambda) = (3+\lambda)^2$$

$$\therefore \lambda_1 = \lambda_2 = -3, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \to v = \begin{pmatrix} v_1 \\ 0 \end{pmatrix}, \quad \forall v_1$$

No other solution since there is only one free parameter.

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Example

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \quad \text{take } v_1 = 1$$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \to p = \begin{pmatrix} p_1 \\ 1 \end{pmatrix} \quad \text{take } p_1 = 1$$

$$v = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad p = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

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Example

$$x' = Ax$$

$$x(t) = c_1 v e^{\lambda t} + c_2 \left(vte^{\lambda t} + pe^{\lambda t}\right)$$

$$= c_1 \begin{pmatrix} e^{-3t} \\ 0 \end{pmatrix} + c_2 \left(\begin{pmatrix} te^{-3t} \\ 0 \end{pmatrix} + \begin{pmatrix} e^{-3t} \\ e^{-3t} \end{pmatrix}\right)$$

is the general solution.

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Example

To solve the IVP we must solve a linear system of algebraic equations.

$$x' = Ax, x(0) = \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix}$$

$$c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix}$$

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Example

$$mu'' + \gamma u' + ku = 0$$
$$x_1 = u, \quad x_1' = x_2 = u'$$

$$x_1' = x_2$$
$$mx_2' + \gamma x_2 + kx_1 = 0$$

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{\gamma}{m} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

If $m = 1, \gamma = 4, k = 4 \rightarrow$ critically damped: $\gamma = 2\sqrt{km}$.

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Example

$$A = \begin{pmatrix} 0 & 1 \\ -4 & -4 \end{pmatrix}$$

$$\begin{vmatrix} -\lambda & 1 \\ -4 & -4-\lambda \end{vmatrix} = -\lambda(-4-\lambda) + 4 = \lambda^2 + 4\lambda + 4 = (\lambda+2)^2$$

$$\lambda_1 = \lambda_2 = -2$$

$$\begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow 2v_1 = -v_2$$

Only one free parameter and one eigenvector direction.

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Example

Take $v_1 = 1$

$$(A + 2I)p = v$$

$$\begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \rightarrow 2p_1 + p_2 = 1$$

$$p_1 = 1 \rightarrow p = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

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Example

$$x(t) = c_1 v e^{\lambda t} + c_2 \left(v t e^{\lambda t} + p e^{\lambda t} \right)$$

$$= c_1 \begin{pmatrix} e^{-2t} \\ -2e^{-2t} \end{pmatrix} + c_2 \left(\begin{pmatrix} te^{-2t} \\ -2te^{-2t} \end{pmatrix} + \begin{pmatrix} e^{-2t} \\ -e^{-2t} \end{pmatrix} \right)$$

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} u(t) \\ u'(t) \end{pmatrix} = \begin{pmatrix} c_1 e^{-2t} + c_2 t e^{-2t} + c_2 e^{-2t} \\ -2c_1 e^{-2t} - 2c_2 t e^{-2t} - c_2 e^{-2t} \end{pmatrix}$$

is the general solution.