Set 23: Systems of First Order Linear Equations Part 4

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Methods for Nonhomogenesous Linear Systems

We have methods to solve the homogeneous linear system of ODEs

$$x'(t) = Ax(t)$$

with constant coefficient matrix A.

We need methods to generate a particular solution for

$$x'(t) = Ax(t) + g(t)$$

Methods

Methods we discussed earlier generalize:

- Method of Undetermined Coefficients
- Variation of Parameters
- Laplace Transforms (See textbook pages 438 and 439)

Additionally, we can use the structure of A to transform the problem into several scalar problems.

Diagonal A

Suppose we have

$$x'(t) = Ax(t) + g(t)$$

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} \alpha_{11} & 0 \\ 0 & \alpha_{22} \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix}$$

Diagonal A

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} \alpha_{11} & 0 \\ 0 & \alpha_{22} \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix}$$

$$x_1'(t) = \alpha_{11} x_1(t) + g_1(t)$$
$$x_2'(t) = \alpha_{22} x_2(t) + g_2(t)$$

Two independent scalar linear constant coefficient nonhomogeneous ODEs.

Diagonal A

- We know how to solve them using techniques discussed earlier.
- Specifically, we can find particular solutions to combine with the homogeneous general solutions determined by eigenvalue analysis.
- This is identical to the combination you get by solving each independently with the general solutions, e.g.:

$$x_i(t) = c_i e^{\alpha_{ii} t} + e^{\alpha_{ii} t} \int e^{-\alpha_{ii} t} g_i(t) dt, \quad i = 1, \ 2$$

- Of course, you can use any method you wish, on the scalar problem and get its essentially equivalent form.
- Therefore, diagonal structure in A means a particular vector solution is simply a vector of particular solutions from independent scalar linear ODEs.

Triangular A

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ 0 & \alpha_{22} \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix}$$

$$x_1'(t) = \alpha_{11} x_1(t) + \alpha_{12} x_2(t) + g_1(t)$$

$$x_2'(t) = \alpha_{22} x_2(t) + g_2(t)$$

Two dependent scalar linear constant coefficient nonhomogeneous ODEs.

Triangular A

Note that the second scalar ODE is independent of the first.

We can solve for a particular solution, $x_2^{(p)}(t)$ of

$$x_2'(t) = \alpha_{22} x_2(t) + g_2(t)$$

We can then substitute and solve for a particular solution, $x_1^{(p)}(t)$ of

$$x_1'(t) = \alpha_{11} x_1(t) + \alpha_{12} x_2^{(p)}(t) + g_1(t) = \alpha_{11} x_1(t) + \tilde{g}(t)$$

General Solution of Structured Problems

To solve $x'(t) = Ax(t) + g(t)$ when A is diagonal or triangular:

- Find eigenvalues and eigenvectors (or generalized eigenvectors) of A.
- Form fundamental solutions

$$x_h'(t) = Ax_h(t) \leftrightarrow x_h(t) = c_1 x^{(1)}(t) + c_2 x^{(2)}(t)$$

- Use diagonal or triangular techniques to find a particular soluion

$$x^{(p)}(t) = \begin{pmatrix} x_1^{(p)}(t) \\ x_2^{(p)}(t) \end{pmatrix}$$

- Form general solution

$$x(t) = x_h(t) + x^{(p)}(t) = c_1 x^{(1)}(t) + c_2 x^{(2)}(t) + x^{(p)}(t)$$

General Solution of Linear Sytems

To solve $x' = Ax + g$ for any A with real eigenvalues we transform the problem. Let T be a nonsingular matrix.

$$x' = Ax + g$$

$$T^{-1}x' = T^{-1}Ax + T^{-1}g$$

$$T^{-1}x' = T^{-1}AIx + T^{-1}g$$

$$T^{-1}x' = T^{-1}A(TT^{-1})x + T^{-1}g$$

$$T^{-1}x' = (T^{-1}AT)(T^{-1}x) + T^{-1}g$$

$$y' = My + f$$

$$Ty = x, \quad f = T^{-1}g, \quad M = (T^{-1}AT)$$

General Solution of Linear Sytems

- M is diagonal if A has two distinct real eigenvalues
- M is upper triangular if A has one repeated eigenvalue with and eigenvector and a generalized eigenvector
- T is determined by the two eigenvectors or an eigenvector and a generalized eigenvector
- We find them when computing the homogeneous general solution.

Example with Distinct Eigenvalues

$$A = \begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix} \rightarrow \lambda_1 = 2, \quad \lambda_2 = -1$$

$$T = \begin{pmatrix} v^{(1)} & v^{(2)} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix}$$

$$T^{-1} = \frac{1}{3} \begin{pmatrix} 4 & -1 \\ -1 & 1 \end{pmatrix}$$

Example with Distinct Eigenvalues

$$A = \begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix}, \quad T^{-1} = \frac{1}{3}\begin{pmatrix} 4 & -1 \\ -1 & 1 \end{pmatrix}$$

$$T^{-1}AT = \frac{1}{3}\begin{pmatrix} 4 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} 4 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 2 & -4 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 6 & 0 \\ 0 & -3 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$$

Example with Repeated Eigenvalue

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \rightarrow \lambda_1 = \lambda_2 = 2$$

$$T = \begin{pmatrix} v & p \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}$$

$$T^{-1} = -\begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}$$

Example with Repeated Eigenvalue

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}, \quad T^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}$$

$$T^{-1}AT = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -2 & -3 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$$

General Solution of Linear Systems

$$x' = Ax + g$$

$$A \to M, \quad T, \quad T^{-1}$$

solve homogeneous system $\quad x' = Ax \to x_h = c_1 x^{(1)} + c_2 x^{(2)}$

$$f = T^{-1} g$$

solve $\quad y' = My + f$

recover particular solution: $\quad x^{(p)} = Ty$

$$x(t) = c_1 x^{(1)} + c_2 x^{(2)} + x^{(p)}$$

Example

$$x' = Ax + g, \quad A = \begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix}, \quad \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} t^2 - 2t \\ 4t^2 - 2t \end{pmatrix}$$

$$T = \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix}, \quad T^{-1} = \frac{1}{3} \begin{pmatrix} 4 & -1 \\ -1 & 1 \end{pmatrix}, \quad M = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 4 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 4g_1 - g_2 \\ -g_1 + g_2 \end{pmatrix} = \begin{pmatrix} -2t \\ t^2 \end{pmatrix}$$

Example

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} e^{2t} \int e^{-2t} f_1 dt \\ e^{-t} \int e^{t} f_2 dt \end{pmatrix}$$

Example

$$x = Ty$$

$$\begin{pmatrix} x_1^{(p)} \\ x_2^{(p)} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \left(e^{2t} \int e^{-2t} f_1 dt\right) + \left(e^{-t} \int e^{t} f_2 dt\right) \\ \left(e^{2t} \int e^{-2t} f_1 dt\right) + 4 \left(e^{-t} \int e^{t} f_2 dt\right) \end{pmatrix}$$

$$x(t) = c_1 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ 4 \end{pmatrix} + \begin{pmatrix} \left(e^{2t} \int e^{-2t} f_1 dt\right) + \left(e^{-t} \int e^{t} f_2 dt\right) \\ \left(e^{2t} \int e^{-2t} f_1 dt\right) + 4 \left(e^{-t} \int e^{t} f_2 dt\right) \end{pmatrix}$$

Example

- We therefore have a general solution that can be adapted to each forcing vector $g(t)$ applied to the system of ODEs.
- For our example of $g_1 = t^2 - 2t$ and $g_2 = 4t^2 - 2t$ yields $f_1 = -2t$ and $f_2 = t^2$
- We have

$$e^{2t} \int e^{-2t}(-2t)dt = e^{2t}(-\frac{1}{2}e^{-2t}(-2t-1)) = t + \frac{1}{2}$$

$$e^{-t} \int e^{t}(t^2)dt = e^{-t}(e^{t}(t^2 - 2t + 2)) = (t^2 - 2t + 2)$$

Example

We therefore have

$$y = \begin{pmatrix} t + \frac{1}{2} \\ t^2 - 2t + 2 \end{pmatrix}$$

$$x^{(p)} = Ty = \begin{pmatrix} y_1 + y_2 \\ y_1 + 4y_2 \end{pmatrix} = \begin{pmatrix} t^2 - t + \frac{5}{2} \\ 4t^2 - 7t + \frac{17}{2} \end{pmatrix}$$

$$x(t) = c_1 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ 4 \end{pmatrix} + \begin{pmatrix} t^2 - t + \frac{5}{2} \\ 4t^2 - 7t + \frac{17}{2} \end{pmatrix}$$

Verification of Solution of Example

Verify that y solves $y' = My + f$:

$$y = \begin{pmatrix} t + \frac{1}{2} \\ t^2 - 2t + 2 \end{pmatrix} \to y' = \begin{pmatrix} 1 \\ 2t - 2 \end{pmatrix}$$

$$My + f = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} t + \frac{1}{2} \\ t^2 - 2t + 2 \end{pmatrix} + \begin{pmatrix} -2t \\ t^2 \end{pmatrix}$$

$$= \begin{pmatrix} 2t + 1 \\ -t^2 + 2t - 2 \end{pmatrix} + \begin{pmatrix} -2t \\ t^2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2t - 2 \end{pmatrix} = y'$$

Verification of Solution of Example

Verify that $x^{(p)}$ solves $x' = Ax + g$:

$$x^{(p)} = \begin{pmatrix} t^2 - t + \frac{5}{2} \\ 4t^2 - 7t + \frac{17}{2} \end{pmatrix} \to \left(x^{(p)}\right)' = \begin{pmatrix} 2t - 1 \\ 8t - 7 \end{pmatrix}$$

$$Ax^{(p)} + g = \begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} t^2 - t + \frac{5}{2} \\ 4t^2 - 7t + \frac{17}{2} \end{pmatrix} + \begin{pmatrix} t^2 - t \\ 4t^2 - 2t \end{pmatrix}$$

$$= \begin{pmatrix} -t^2 + 4t - 1 \\ -4t^2 + 10t - 7 \end{pmatrix} + \begin{pmatrix} t^2 - 2t \\ 4t^2 - 2t \end{pmatrix} = \begin{pmatrix} 2t - 1 \\ 8t - 7 \end{pmatrix} = \left(x^{(p)}\right)'$$

Triangular Example

- If A has a repeated eigenvalue, λ, with only one eigenvector then

$$T^{-1}AT = M = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

- Solve for a particular solution, $y_2^{(p)}$, of the second scalar linear ODE governing y_2.
- Solve for a particular solution, $y_1^{(p)}$, of the first scalar linear ODE governing y_1 using $y_2^{(p)}$.
- Transform the solution back to $x^{(p)}$.