Set 2: First Order ODEs - Part 1

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Ordinary Differential Equations
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First Order ODE Initial Value Problem

- $t \in \mathbb{R} \mapsto y(t) \in \mathbb{R}$
- $(t, y) \in \mathbb{R}^2 \mapsto f(t, y) \in \mathbb{R}$
- $y'(t)$ is the highest order of derivative involved
- The initial value problem:

$$y' = f(t, y), \quad y(t_0) = y_0$$

- no general solution so we consider subclasses:
  - linear equations
  - separable equations
  - exact equations

Undetermined Coefficients

The Undetermined Coefficients Method:

- hypothesizes a form of the solution that is a function of a set of coefficients
- substitutes the form of the solution into the differential equation
- solves for the unknown coefficients if possible

This will be used later on more complicated problems but we apply it to a simple problem here.

Undetermined Coefficients

Consider the general first order linear constant-coefficient ODE

$$y' = ay - b \tag{1}$$

where $a \neq 0$ and b are real constants.

- if $b = 0$ we know $y_1' = ay_1 \rightarrow y_1(t) = Ce^{at}$
- Equation (1) adds a constant to y_1' to get y'
- Equation (1) says y' proportional to y
- Hypothesize that there is a constant k such that

$$y(t) = y_1(t) + k$$

Undetermined Coefficients

- substitute $y(t) = y_1(t) + k$ into ODE RHS :

$$ay - b = ay_1(t) + ak - b = aCe^{at} + ak - b$$

- substitute $y(t) = y_1(t) + k$ into ODE LHS :

$$\frac{dy}{dt} = \frac{d}{dt}\big(y_1(t) + k\big) = \frac{d}{dt}\big(Ce^{at}\big) + \frac{d}{dt}(k) = aCe^{at}$$

- equate RHS and LHS to get conditions on k and check for consistency with assumptions:

$$\text{if} \quad aCe^{at} = aCe^{at} + ak - b \quad \text{then} \quad 0 = ak - b$$

$$k = \frac{b}{a}$$

Undetermined Coefficients

- Since $a \neq 0$, k is a well-defined constant.
- Consistent with assumptions on solution, i.e., constant k.
- The function

$$y(t) = Ce^{at} + \frac{b}{a}$$

should be the general solution to the ODE.

- Verify:

$$y' = \frac{d}{dt}\big(Ce^{at} + \frac{b}{a}\big) = aCe^{at}$$

$$ay - b = a\big(Ce^{at} + \frac{b}{a}\big) - b = aCe^{at} + b - b = aCe^{at}$$

Undetermined Coefficients

If the correct form of $y(t)$ was not the hypothesis we should get a contradiction.

For example, hypothesize that there is a constant k such that

$$y(t) = y_1(t) + kt = Ce^{at} + kt$$

LHS: $\frac{d}{dt}\big(Ce^{at} + kt\big) = aCe^{at} + k$

RHS: $ay - b = aCe^{at} + akt - b$

if LHS=RHS then $aCe^{at} + k = aCe^{at} + akt - b$

$$b = k(at - 1) \rightarrow k = \frac{b}{at - 1}$$

This is a contradiction to the assumption that k is constant.

Summary

First order linear constant coefficient ODE:

$$y' = ay - b$$

where $a \in \mathbb{R}, \quad a \neq 0$ and $b \in \mathbb{R}$

$$y(t) = Ce^{at} + \frac{b}{a} \qquad \text{general solution}$$

$$y(t) = \left\{e^{-at_0}\big(y_0 - \frac{b}{a}\big)\right\} e^{at} + \frac{b}{a} \qquad \text{IVP solution}$$

for $y(t_0) = y_0$

Integrating Factor Method

Consider the extension to a slightly more complicated first order linear ODE:

$$y' + ay = g(t)$$

where $t \in \mathbb{R} \mapsto g(t) \in \mathbb{R}$ is a continuous function and $a \in \mathbb{R}$ is a constant.

An integrating factor $\mu(t)$ is a function so that the solution of the ODE

$$\mu(t)y' + \mu(t)ay = \mu(t)g(t) \qquad (2)$$

can be found "easily".

Basic Facts

Recall the product rule for derivatives:

$$\frac{d}{dt}(rs) = \frac{dr}{dt}s + r\frac{ds}{dt}$$

$$(rs)' = r's + rs'$$

This yields integration by parts:

$$\int rs'dt = rs - \int r's \;\; dt + c$$

Integrating Factor

Compare LHS of the ODE containing $\mu(t)$ and the product rule:

$$\text{LHS:} \quad \mu(t)y' + \mu(t)ay$$

$$\text{PR:} \quad \mu(t)y' + \mu'(t)y$$

They are equivalent if

$$\mu' = a\mu$$

Integrating Factor

We know the solution for the ODE governing $\mu(t)$:

$$\mu' = a\mu \rightarrow \mu(t) = Ce^{at}$$

We can choose any C and for now we choose $C = 1$

$$\therefore \mu(t) = e^{at}$$

is our integrating factor for ODE (2)

Consequence of Use of Integrating Factor

$$\mu(t)y' + \mu(t)ay = \mu(t)g(t)$$
$$(e^{at})y' + (e^{at}a)y = e^{at}g(t)$$
$$(e^{at})y' + (e^{at})'y = (e^{at}y)' = e^{at}g(t)$$
$$\therefore e^{at}y = \int e^{at}g(t)\,dt + C$$

So if $\int e^{at}g(t)\,dt$ can be found we have a solution to ODE (2).

$$y(t) = Ce^{-at} + e^{-at}\int e^{at}g(t)\,dt$$

Consistency

If $g(t) = -b$ we should have the solution derived earlier:

$$y(t) = Ce^{-at} + e^{-at}\int e^{at}g(t)\,dt$$
$$= Ce^{-at} - be^{-at}\int e^{at}\,dt$$
$$= Ce^{-at} - be^{-at}\frac{e^{at}}{a}$$
$$= Ce^{-at} - \frac{b}{a}$$

This is consistent with earlier solution of $y' = -ay - b$.

Example

$$y' - 2y = 4 - t$$
$$\mu(t)y' - 2\mu(t)y = \mu(t)y' + \mu'(t)y = \mu(t)(4-t)$$

$$\mu' = -2\mu \rightarrow \mu(t) = e^{-2t}$$

$$(e^{-2t}y)' = e^{-2t}(4-t)$$
$$e^{-2t}y = C + \int e^{-2t}(4-t)\,dt$$
$$e^{-2t}y = C + 4\int e^{-2t}\,dt - \int te^{-2t}\,dt$$

Example

$$e^{-2t}y = C + 4\int e^{-2t}\,dt - \int te^{-2t}\,dt$$

$$4\int e^{-2t}\,dt = -2\int e^{z}\,dz = -2e^{z} = -2e^{-2t}$$

$$\therefore e^{-2t}y = C - 2e^{-2t} - \int te^{-2t}\,dt$$

Example

$$\int rs'dt = rs - \int r's\,dt + c$$

$$r = t \quad \text{and} \quad s' = e^{-2t}$$

$$r' = 1 \quad \text{and} \quad s = -\frac{1}{2}e^{-2t}$$

$$\int te^{-2t}\,dt = -\frac{1}{2}te^{-2t} + \frac{1}{2}\int e^{-2t}\,dt$$
$$= -\frac{1}{2}te^{-2t} - \frac{1}{4}e^{-2t} + c$$

Example

$$e^{-2t}y = C - 2e^{-2t} - \int te^{-2t}\,dt$$
$$= C - 2e^{-2t} + \frac{1}{2}te^{-2t} + \frac{1}{4}e^{-2t}$$

$$\therefore y(t) = Ce^{2t} - 2 + \frac{1}{2}t + \frac{1}{4} = Ce^{2t} + \frac{1}{2}t - \frac{7}{4}$$

Example

Verify solution:

$$y(t) = Ce^{2t} + \frac{1}{2}t - \frac{7}{4}$$

$$y' - 2y = \left(2Ce^{2t} + \frac{1}{2}\right) - 2\left(Ce^{2t} + \frac{1}{2}t - \frac{7}{4}\right)$$
$$= 2Ce^{2t} + \frac{1}{2} - 2Ce^{2t} - t + \frac{7}{2}$$
$$= 4 - t$$

Analysis of Behavior of Solution

Consider $t \geq 0$,

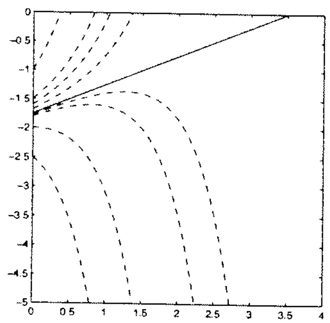
$$y(t) = Ce^{2t} + \frac{1}{2}t - \frac{7}{4}$$

$$y(0) = -\frac{7}{4} \rightarrow C = 0, \quad \therefore y(t) = \frac{1}{2}t - \frac{7}{4}$$

$$y(0) > -\frac{7}{4} \rightarrow C > 0, \quad \therefore y(t) \rightarrow \infty \quad \text{exponentially}$$

$$y(0) < -\frac{7}{4} \rightarrow C < 0, \quad \therefore y(t) \rightarrow -\infty \quad \text{exponentially}$$

Example: Integral Curves



Integral curves for $y' - 2y = 4 - t$

Summary of Solutions

$$y' = b \longrightarrow y(t) = bt + C$$

$$y' = ay \longrightarrow y(t) = Ce^{at}$$

$$y' = ay - b \longrightarrow y(t) = Ce^{at} + \frac{b}{a}$$

$$y' + ay = g(t) \longrightarrow y(t) = Ce^{-at} + e^{-at} \int e^{at} g(t)\, dt$$

General Linear Form

The general explicit first order linear ODE can be written

$$y' + p(t)y = g(t) \tag{3}$$

The integrating factor method derivation for $p(t) = a$ is easily adapted to find a general solution:

$$\text{if} \quad \mu' = p(t)\mu \quad \text{then}$$

$$\mu(t)g(t) = \mu(t)y' + \mu(t)p(t)y = \mu(t)y' + \mu(t)'y = \left(\mu(t)y(t)\right)'$$

$$\therefore \mu(t)y = C + \int \mu(t)g(t)\, dt$$

General Linear Form

Hypothesize based on $p(t) = a$ form

$$\mu(t) = \tilde{C}e^{z(t)}, \quad z(t) = \int p(t)\, dt$$

$$\mu'(t) = \tilde{C}e^{z(t)}z'(t) = \tilde{C}e^{z(t)}p(t) = \mu(t)p(t)$$

$$y(t) = \frac{C}{\mu(t)} + \frac{1}{\mu(t)} \int \mu(t)g(t)\, dt$$

Note two integrations needed to solve the problem.

General Linear Form

For an IVP we have $y(t_0) = y_0$. Assume $\tilde{C}$ is such that

$$\mu(t_0) = 1$$

We have

$$y(t) = \frac{C}{\mu(t)} + \frac{1}{\mu(t)} \int \mu(t)g(t)\,dt = \frac{C}{\mu(t)} + \frac{1}{\mu(t)} \int_{t_0}^{t} \mu(s)g(s)\,ds$$

$$y_0 = y(t_0) = \frac{C}{\mu(t_0)} + 0 \longrightarrow C = y_0$$

$$y(t) = \frac{1}{\mu(t)} \left\{ y_0 + \int_{t_0}^{t} \mu(s)g(s)\,ds \right\}$$

Example

$$t \geq 1, \quad ty' + 2y = 4t^2 \rightarrow y' + \frac{2}{t}y = 4t$$

$$\int p(t)\,dt = 2\int \frac{1}{t}\,dt = 2\ln|t|$$

$$\mu(t) = \tilde{C}e^{2\ln|t|} = \tilde{C}|t|^2 = \tilde{C}t^2$$

$$(t^2y)' = t^2y' + 2ty = 4t^3$$

$$t^2y = t^4 + C \longrightarrow y(t) = t^2 + \frac{C}{t^2}$$

$$y_1 = y(1) = 1 + C \longrightarrow C = y_1 - 1$$

$$y(t) = t^2 + \frac{y_1 - 1}{t^2}$$

Analysis of Behavior of Solution

We have

$$y(t) = t^2 + \frac{C}{t^2} = t^2 + \frac{y_1 - 1}{t^2}, \quad t \geq 1$$

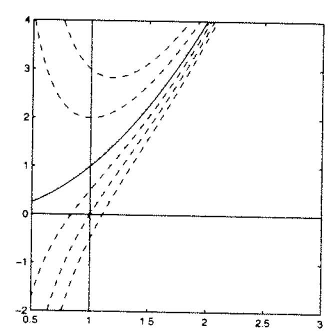
if $y_1 = 1$ then $C = 0$ and $y(t) = t^2$

if $y_1 > 1$ then $C > 0$ and $y(t) \rightarrow t^2$ from above

if $y_1 < 1$ then $C < 0$ and $y(t) \rightarrow t^2$ from below

Note what happens as $t_0 \rightarrow 0$.

Example: Integral Curves



Integral curves for $y' + \frac{2}{t}y = 4t$ and $t \geq 1$