Set 3: First Order ODEs - Part 2

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Ordinary Differential Equations
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Problem 3.1

Textbook, p. 39, Problem 1

$$y' + 3y = t + e^{-2t}$$

$$p(t) = 3, \quad g(t) = t + e^{-2t}$$

$$\text{if} \quad \mu'(t) = p(t)\mu(t) = 3\mu(t) \quad \text{then} \quad \mu(t) = e^{3t}$$

$$\text{and} \quad (\mu(t)y)' = \mu(t)y' + \mu'(t)y = \mu(t)g(t) = e^{3t}(t + e^{-2t})$$

$$y(t) = C\mu^{-1}(t) + \mu^{-1}(t) \int \mu(t)g(t)\, dt$$

Problem 3.1

Textbook, p. 39, Problem 1

$$y(t) = C\mu^{-1}(t) + \mu^{-1}(t) \int \mu(t)g(t)\ dt$$

$$y(t) = Ce^{-3t} + e^{-3t} \int e^{3t}(t + e^{-2t})\ dt$$

$$= Ce^{-3t} + e^{-3t}\big(e^{t} + \int te^{3t}dt\big)$$

use integration by parts $\int rs' = rs - \int r's$

with $r = t \quad r' = 1, \quad s' = e^{3t}, \quad s = \frac{1}{3}e^{3t}$

$$\int te^{3t}dt = \frac{1}{3}te^{3t} - \int \frac{1}{3}e^{3t}dt = \frac{1}{3}te^{3t} - \frac{1}{9}e^{3t}$$

Problem 3.1

Textbook, p. 39, Problem 1

$$\int te^{3t}dt = \frac{1}{3}te^{3t} - \frac{1}{9}e^{3t}$$

$$y(t) = Ce^{-3t} + e^{-3t}\left(e^{t} + \int te^{3t}dt\right)$$

$$y(t) = Ce^{-3t} + e^{-2t} + \frac{1}{3}t - \frac{1}{9}$$

Exponentials go to 0 for any initial condition and $y(t) \to \frac{1}{3}t - \frac{1}{9}$.

Problem 3.2

Textbook, p. 39, Problem 11

$$y' + y = 5(\sin 2t)$$

$$p(t) = 1, \quad g(t) = 5(\sin 2t)$$

$$\text{if} \quad \mu'(t) = p(t)\mu(t) = \mu(t) \quad \text{then} \quad \mu(t) = e^t$$

$$\text{and} \quad (\mu(t)y)' = \mu(t)y' + \mu'(t)y = \mu(t)g(t) = e^t(5(\sin 2t))$$

$$y(t) = C\mu^{-1}(t) + \mu^{-1}(t)\int \mu(t)g(t)\, dt$$

Problem 3.2

Textbook, p. 39, Problem 11

$$y(t) = C\mu^{-1}(t) + \mu^{-1}(t) \int \mu(t) g(t)\; dt$$

$$y(t) = Ce^{-t} + 5e^{-t} \int e^{t} (\sin 2t)\; dt$$

$$\text{recall} \quad \int e^{au} (\sin bu)\; du = \frac{e^{au}}{a^2 + b^2} \big(a(\sin bu) - b(\cos bu)\big)$$

$$\therefore y(t) = Ce^{-t} + e^{-t}\big[e^{t}(\sin 2t) - 2e^{t}(\cos 2t)\big]$$

$$= Ce^{-t} + (\sin 2t) - 2(\cos 2t)$$

The exponential goes to 0 for any initial condition and and $y(t) \to (\sin 2t) - 2(\cos 2t)$ as $t \to \infty$.

Problem 3.2

Textbook, p. 39, Problem 11

Note rather than use the general form of $\int e^{au}(\sin bu)\ du$ one can derive the specific form for $\int e^t(\sin 2t)\ dt$ as follows:

$$\text{use integration by parts} \quad \int rs' = rs - \int r's$$

$$\text{with} \quad r = \sin 2t \quad r' = 2\cos 2t, \quad s' = e^t, \quad s = e^t$$

$$\int e^t(\sin 2t)\ dt = e^t(\sin 2t) - 2\int e^t(\cos 2t)\ dt$$

Problem 3.2

to get $$\int e^t(\cos 2t)\,dt$$

use integration by parts $$\int rs' = rs - \int r's$$

with $r = \cos 2t \quad r' = -2\sin 2t, \quad s' = e^t, \quad s = e^t$

$$\int e^t(\cos 2t)\,dt = e^t(\cos 2t) + 2\int e^t(\sin 2t)\,dt$$

Note repetition of previous integral.

Problem 3.2

Textbook, p. 39, Problem 11

Putting it all together yields:

$$\int e^t(\cos 2t)\ dt = e^t(\cos 2t) + 2\int e^t(\sin 2t)\ dt$$

$$\int e^t(\sin 2t)\ dt = e^t(\sin 2t) - 2\big(e^t(\cos 2t) + 2\int e^t(\sin 2t)\ dt\big)$$

$$= e^t(\sin 2t) - 2e^t(\cos 2t) - 4\int e^t(\sin 2t)\ dt$$

$$5\int e^t(\sin 2t)\ dt = e^t(\sin 2t) - 2e^t(\cos 2t)$$

$$\int e^t(\sin 2t)\ dt = \frac{1}{5}\big(e^t(\sin 2t) - 2e^t(\cos 2t)\big)$$

as desired.

Problem 3.3

Textbook, p. 39, Problem 12

$$2y' + y = 3t^2$$

$$y' + \frac{1}{2}y = \frac{3}{2}t^2$$

$$p(t) = 0.5, \quad g(t) = \frac{3}{2}t^2$$

$$\text{if} \quad \mu'(t) = p(t)\mu(t) = \mu(t) \quad \text{then} \quad \mu(t) = e^{0.5t}$$

$$\text{and} \quad (\mu(t)y)' = \mu(t)y' + \mu'(t)y = \mu(t)g(t) = e^{0.5t}(\frac{3}{2}t^2)$$

$$y(t) = C\mu^{-1}(t) + \mu^{-1}(t)\int \mu(t)g(t)\, dt$$

Problem 3.3

Textbook, p. 39, Problem 12

$$y(t) = C\mu^{-1}(t) + \mu^{-1}(t) \int \mu(t) g(t)\, dt$$

$$y(t) = Ce^{-0.5t} + e^{-0.5t} \int e^{0.5t} (\frac{3}{2} t^2)\, dt$$

$$= Ce^{-0.5t} + \frac{3}{2} e^{-0.5t} \int t^2 e^{0.5t}\, dt$$

use integration by parts $\int rs' = rs - \int r's$

with $r = t^2 \quad r' = 2t, \quad s' = e^{0.5t}, \quad s = 2e^{0.5t}$

$$\int t^2 e^{0.5t}\, dt = 2t^2 e^{0.5t} - 4 \int t e^{0.5t}\, dt$$

Problem 3.3

Textbook, p. 39, Problem 12

$$\text{We have} \quad \int t^2 e^{0.5t}\,dt = 2t^2 e^{0.5t} - 4\int te^{0.5t}\,dt$$

$$\text{integrate by parts again} \quad r = t \quad r' = 1, \quad s' = e^{0.5t}, \quad s = 2e^{0.5t}$$

$$\int te^{0.5t}\,dt = 2te^{0.5t} - 2\int e^{0.5t} = 2te^{0.5t} - 4e^{0.5t}$$

$$\therefore \int t^2 e^{0.5t}\,dt = 2t^2 e^{0.5t} - 4\big(2te^{0.5t} - 4e^{0.5t}\big)$$

$$= 2t^2 e^{0.5t} - 8te^{0.5t} + 16e^{0.5t}$$

Problem 3.3

Textbook, p. 39, Problem 12

$$\int t^2 e^{0.5t}\, dt = 2t^2 e^{0.5t} - 8te^{0.5t} + 16e^{0.5t}$$

$$y(t) = Ce^{-0.5t} + \frac{3}{2}e^{-0.5t} \int t^2 e^{0.5t}\, dt$$

$$\therefore y(t) = Ce^{-0.5t} + \frac{3}{2}e^{-0.5t}\left(2t^2 e^{0.5t} - 8te^{0.5t} + 16e^{0.5t}\right)$$

$$= Ce^{-0.5t} + \frac{3}{2}\left(2t^2 - 8t + 16\right) = Ce^{-0.5t} + 3t^2 - 12t + 24$$

The exponential goes to 0 for any initial condition and and $y(t) \to 3t^2 - 12t + 24$ as $t \to \infty$.

Problem 3.4

Textbook, p. 39, Problem 4

$$y' + t^{-1}y = 3(\cos 2t), \quad t > 0$$

$$p(t) = t^{-1}, \quad g(t) = 3(\cos 2t)$$

$$\text{if} \quad \mu'(t) = p(t)\mu(t) = t^{-1}\mu(t) \quad \text{then}$$

$$\int p(t)dt = \int \frac{1}{t}\, dt = \ln|t|,$$

$$\mu(t) = e^{\int p(t)dt} = e^{\ln|t|} = t$$

$$\text{and} \quad (\mu(t)y)' = \mu(t)y' + \mu'(t)y = \mu(t)g(t) = t(3(\cos 2t))$$

$$y(t) = C\mu^{-1}(t) + \mu^{-1}(t)\int \mu(t)g(t)\, dt$$

Problem 3.4

Textbook, p. 39, Problem 4

$$y(t) = C\mu^{-1}(t) + \mu^{-1}(t)\int \mu(t)g(t)\,dt$$

$$y(t) = Ct^{-1} + 3t^{-1}\int t(\cos 2t)\,dt$$

use integration by parts $\int rs' = rs - \int r's$

with $r = t \quad r' = 1, \quad s' = (\cos 2t), \quad s = 0.5(\sin 2t)$

$$\int t((\cos 2t))dt = 0.5t(\sin 2t) - 0.5\int(\sin 2t)$$

$$= 0.5t(\sin 2t) + 0.25(\cos 2t)$$

Problem 3.4

Textbook, p. 39, Problem 4

$$y(t) = C\mu^{-1}(t) + \mu^{-1}(t) \int \mu(t)g(t)\ dt$$

$$y(t) = Ct^{-1} + 3t^{-1} \int t(\cos 2t)\ dt$$

$$y(t) = Ct^{-1} + 3t^{-1}\big(0.5t(\sin 2t) + 0.25(\cos 2t)\big)$$

$$= Ct^{-1} + \frac{3}{2}(\sin 2t) + \frac{3}{4}\frac{(\cos 2t)}{t}$$

$Ct^{-1} \to 0$ and $\cos 2t$ is bounded.

$$\therefore y(t) \to \frac{3}{2}(\sin 2t) \text{as} \ \ t \to \infty$$

Problem 3.5

Textbook, p. 39, Problem 8

$$(1+t^2)y' + 4ty = (1+t^2)^{-2} \to y' + 4t(1+t^2)^{-1}y = (1+t^2)^{-3}$$

$$p(t) = 4t(1+t^2)^{-1}, \quad g(t) = (1+t^2)^{-3}$$

$$\text{if} \quad \mu'(t) = p(t)\mu(t) = 4t(1+t^2)^{-1}\mu(t) \quad \text{then}$$

$$\int p(t)\,dt = 2\int \frac{2t}{(1+t^2)}\,dt = 2\int \frac{1}{z}\,dz = 2\ln|1+t^2| = 2\ln(1+t^2)$$

$$\mu(t) = e^{\int p(t)dt} = e^{2\ln(1+t^2)} = (1+t^2)^2$$

$$\text{and} \quad (\mu(t)y)' = \mu(t)y' + \mu'(t)y = \mu(t)g(t) = \frac{1}{(1+t^2)}$$

Problem 3.5

Textbook, p. 39, Problem 8

$$y(t) = C\mu^{-1}(t) + \mu^{-1}(t) \int \mu(t) g(t) \, dt$$

$$\begin{aligned} y(t) &= \frac{C}{(1+t^2)^2} + \frac{1}{(1+t^2)^2} \int \frac{1}{(1+t^2)} \, dt \\ &= \frac{C}{(1+t^2)^2} + \frac{\tan^{-1} t}{(1+t^2)^2} \end{aligned}$$

The first term goes to 0 for any initial condition. Since $|\tan^{-1} t| \leq \pi/2$ the second term also goes to 0.

Integrating Factor Summary

$$y' + ay = 0$$

$$\Downarrow$$

$$y' + ay = g(t)$$

$$\Downarrow$$

$$y' + p(t)y = g(t)$$

$$\Downarrow$$

general solution: $y(t) = \frac{1}{\mu(t)}\left[C + \int \mu(t)g(t)\ dt\right], \quad \mu(t) = \int p(t)\ dt$

IVP solution: $y(t) = \frac{1}{\mu(t)}\left[y_0 + \int_{t_0}^{t} \mu(s)g(s)\ ds\right], \quad \mu(t_0) = 1$