

Set 3: First Order ODEs - Part 2

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Ordinary Differential Equations

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Problem 3.1

Textbook, p. 39, Problem 1

$$y(t) = C\mu^{-1}(t) + \mu^{-1}(t) \int \mu(t)g(t) dt$$

$$y(t) = Ce^{-3t} + e^{-3t} \int e^{3t}(t + e^{-2t}) dt$$

$$= Ce^{-3t} + e^{-3t} (e^t + \int te^{3t} dt)$$

use integration by parts $\int rs' = rs - \int r's$

with $r = t$ $r' = 1$, $s' = e^{3t}$, $s = \frac{1}{3}e^{3t}$

$$\int te^{3t} dt = \frac{1}{3}te^{3t} - \int \frac{1}{3}e^{3t} dt = \frac{1}{3}te^{3t} - \frac{1}{9}e^{3t}$$

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Problem 3.1

Textbook, p. 39, Problem 1

$$y' + 3y = t + e^{-2t}$$

$$p(t) = 3, \quad g(t) = t + e^{-2t}$$

if $\mu'(t) = p(t)\mu(t) = 3\mu(t)$ then $\mu(t) = e^{3t}$

and $(\mu(t)y)' = \mu(t)y' + \mu'(t)y = \mu(t)g(t) = e^{3t}(t + e^{-2t})$

$$y(t) = C\mu^{-1}(t) + \mu^{-1}(t) \int \mu(t)g(t) dt$$

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Problem 3.1

Textbook, p. 39, Problem 1

$$\int te^{3t} dt = \frac{1}{3}te^{3t} - \frac{1}{9}e^{3t}$$

$$y(t) = Ce^{-3t} + e^{-3t} (e^t + \int te^{3t} dt)$$

$$y(t) = Ce^{-3t} + e^{-2t} + \frac{1}{3}t - \frac{1}{9}$$

Exponentials go to 0 for any initial condition and $y(t) \rightarrow \frac{1}{3}t - \frac{1}{9}$.

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Problem 3.2

Textbook, p. 39, Problem 11

$$y' + y = 5(\sin 2t)$$

$$p(t) = 1, \quad g(t) = 5(\sin 2t)$$

if $\mu'(t) = p(t)\mu(t) = \mu(t)$ then $\mu(t) = e^t$ and $(\mu(t)y)' = \mu(t)y' + \mu'(t)y = \mu(t)g(t) = e^t(5(\sin 2t))$

$$y(t) = C\mu^{-1}(t) + \mu^{-1}(t) \int \mu(t)g(t) dt$$

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Problem 3.2

Textbook, p. 39, Problem 11

$$y(t) = C\mu^{-1}(t) + \mu^{-1}(t) \int \mu(t)g(t) dt$$

$$y(t) = Ce^{-t} + 5e^{-t} \int e^t(\sin 2t) dt$$

$$\text{recall} \int e^{au}(\sin bu) du = \frac{e^{au}}{a^2 + b^2} (a(\sin bu) - b(\cos bu))$$

$$\begin{aligned} \therefore y(t) &= Ce^{-t} + e^{-t} [e^t(\sin 2t) - 2e^t(\cos 2t)] \\ &= Ce^{-t} + (\sin 2t) - 2(\cos 2t) \end{aligned}$$

The exponential goes to 0 for any initial condition and and $y(t) \rightarrow (\sin 2t) - 2(\cos 2t)$ as $t \rightarrow \infty$.

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Problem 3.2

Textbook, p. 39, Problem 11

Note rather than use the general form of $\int e^{au}(\sin bu) du$ one can derive the specific form for $\int e^t(\sin 2t) dt$ as follows:

$$\text{use integration by parts} \quad \int rs' = rs - \int r's$$

$$\text{with } r = \sin 2t \quad r' = 2 \cos 2t, \quad s' = e^t, \quad s = e^t$$

$$\int e^t(\sin 2t) dt = e^t(\sin 2t) - 2 \int e^t(\cos 2t) dt$$

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Problem 3.2

$$\text{to get} \quad \int e^t(\cos 2t) dt$$

$$\text{use integration by parts} \quad \int rs' = rs - \int r's$$

$$\text{with } r = \cos 2t \quad r' = -2 \sin 2t, \quad s' = e^t, \quad s = e^t$$

$$\int e^t(\cos 2t) dt = e^t(\cos 2t) + 2 \int e^t(\sin 2t) dt$$

Note repetition of previous integral.

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Problem 3.2

Textbook, p. 39, Problem 11

Putting it all together yields:

$$\begin{aligned} \int e^t (\cos 2t) dt &= e^t (\cos 2t) + 2 \int e^t (\sin 2t) dt \\ \int e^t (\sin 2t) dt &= e^t (\sin 2t) - 2(e^t (\cos 2t) + 2 \int e^t (\sin 2t) dt) \\ &= e^t (\sin 2t) - 2e^t (\cos 2t) - 4 \int e^t (\sin 2t) dt \\ 5 \int e^t (\sin 2t) dt &= e^t (\sin 2t) - 2e^t (\cos 2t) \\ \int e^t (\sin 2t) dt &= \frac{1}{5} (e^t (\sin 2t) - 2e^t (\cos 2t)) \end{aligned}$$

as desired.

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Problem 3.3

Textbook, p. 39, Problem 12

$$\begin{aligned} 2y' + y &= 3t^2 \\ y' + \frac{1}{2}y &= \frac{3}{2}t^2 \end{aligned}$$

$$p(t) = 0.5, \quad g(t) = \frac{3}{2}t^2$$

if $\mu'(t) = p(t)\mu(t) = \mu(t)$ then $\mu(t) = e^{0.5t}$ and $(\mu(t)y)' = \mu(t)y' + \mu'(t)y = \mu(t)g(t) = e^{0.5t}(\frac{3}{2}t^2)$

$$y(t) = C\mu^{-1}(t) + \mu^{-1}(t) \int \mu(t)g(t) dt$$

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Problem 3.3

Textbook, p. 39, Problem 12

$$\begin{aligned} y(t) &= C\mu^{-1}(t) + \mu^{-1}(t) \int \mu(t)g(t) dt \\ y(t) &= Ce^{-0.5t} + e^{-0.5t} \int e^{0.5t} (\frac{3}{2}t^2) dt \\ &= Ce^{-0.5t} + \frac{3}{2}e^{-0.5t} \int t^2 e^{0.5t} dt \end{aligned}$$

use integration by parts $\int r s' = r s - \int r' s$ with $r = t^2$, $r' = 2t$, $s' = e^{0.5t}$, $s = 2e^{0.5t}$

$$\int t^2 e^{0.5t} dt = 2t^2 e^{0.5t} - 4 \int t e^{0.5t} dt$$

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Problem 3.3

Textbook, p. 39, Problem 12

$$\text{We have } \int t^2 e^{0.5t} dt = 2t^2 e^{0.5t} - 4 \int t e^{0.5t} dt$$

integrate by parts again $r = t$, $r' = 1$, $s' = e^{0.5t}$, $s = 2e^{0.5t}$

$$\int t e^{0.5t} dt = 2t e^{0.5t} - 2 \int e^{0.5t} dt = 2t e^{0.5t} - 4e^{0.5t}$$

$$\begin{aligned} \therefore \int t^2 e^{0.5t} dt &= 2t^2 e^{0.5t} - 4(2t e^{0.5t} - 4e^{0.5t}) \\ &= 2t^2 e^{0.5t} - 8t e^{0.5t} + 16e^{0.5t} \end{aligned}$$

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Problem 3.3

Textbook, p. 39, Problem 12

$$\int t^2 e^{0.5t} dt = 2t^2 e^{0.5t} - 8t e^{0.5t} + 16e^{0.5t}$$

$$y(t) = Ce^{-0.5t} + \frac{3}{2}e^{-0.5t} \int t^2 e^{0.5t} dt$$

$$\begin{aligned} \therefore y(t) &= Ce^{-0.5t} + \frac{3}{2}e^{-0.5t} (2t^2 e^{0.5t} - 8t e^{0.5t} + 16e^{0.5t}) \\ &= Ce^{-0.5t} + \frac{3}{2}(2t^2 - 8t + 16) = Ce^{-0.5t} + 3t^2 - 12t + 24 \end{aligned}$$

The exponential goes to 0 for any initial condition and and $y(t) \rightarrow 3t^2 - 12t + 24$ as $t \rightarrow \infty$.

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Problem 3.4

Textbook, p. 39, Problem 4

$$y' + t^{-1}y = 3(\cos 2t), \quad t > 0$$

$$p(t) = t^{-1}, \quad g(t) = 3(\cos 2t)$$

if $\mu'(t) = p(t)\mu(t) = t^{-1}\mu(t)$ then

$$\int p(t)dt = \int \frac{1}{t} dt = \ln|t|,$$

$$\mu(t) = e^{\int p(t)dt} = e^{\ln|t|} = t$$

and $(\mu(t)y)' = \mu(t)y' + \mu'(t)y = \mu(t)g(t) = t(3(\cos 2t))$

$$y(t) = Ct\mu^{-1}(t) + \mu^{-1}(t) \int \mu(t)g(t) dt$$

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Problem 3.4

Textbook, p. 39, Problem 4

$$y(t) = C\mu^{-1}(t) + \mu^{-1}(t) \int \mu(t)g(t) dt$$

$$y(t) = Ct^{-1} + 3t^{-1} \int t(\cos 2t) dt$$

use integration by parts $\int r s' = r s - \int r' s$

with $r = t$ $r' = 1$, $s' = (\cos 2t)$, $s = 0.5(\sin 2t)$

$$\begin{aligned} \int t((\cos 2t))dt &= 0.5t(\sin 2t) - 0.5 \int (\sin 2t) \\ &= 0.5t(\sin 2t) + 0.25(\cos 2t) \end{aligned}$$

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Problem 3.4

Textbook, p. 39, Problem 4

$$y(t) = C\mu^{-1}(t) + \mu^{-1}(t) \int \mu(t)g(t) dt$$

$$y(t) = Ct^{-1} + 3t^{-1} \int t(\cos 2t) dt$$

$$y(t) = Ct^{-1} + 3t^{-1} (0.5t(\sin 2t) + 0.25(\cos 2t))$$

$$= Ct^{-1} + \frac{3}{2}(\sin 2t) + \frac{3}{4} \frac{(\cos 2t)}{t}$$

$Ct^{-1} \rightarrow 0$ and $\cos 2t$ is bounded.

$$\therefore y(t) \rightarrow \frac{3}{2}(\sin 2t) \text{ as } t \rightarrow \infty$$

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Problem 3.5

Textbook, p. 39, Problem 8

$$(1 + t^2)y' + 4ty = (1 + t^2)^{-2} \rightarrow y' + 4t(1 + t^2)^{-1}y = (1 + t^2)^{-3}$$

$$p(t) = 4t(1 + t^2)^{-1}, \quad g(t) = (1 + t^2)^{-3}$$

if $\mu'(t) = p(t)\mu(t) = 4t(1 + t^2)^{-1}\mu(t)$ then

$$\int p(t) dt = 2 \int \frac{2t}{(1 + t^2)} dt = 2 \int \frac{1}{z} dz = 2 \ln|1 + t^2| = 2 \ln(1 + t^2)$$

$$\mu(t) = e^{\int p(t)dt} = e^{2 \ln(1+t^2)} = (1 + t^2)^2$$

$$\text{and } (\mu(t)y)' = \mu(t)y' + \mu'(t)y = \mu(t)g(t) = \frac{1}{(1 + t^2)}$$

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Problem 3.5

Textbook, p. 39, Problem 8

$$y(t) = C\mu^{-1}(t) + \mu^{-1}(t) \int \mu(t)g(t) dt$$

$$\begin{aligned} y(t) &= \frac{C}{(1 + t^2)^2} + \frac{1}{(1 + t^2)^2} \int \frac{1}{(1 + t^2)} dt \\ &= \frac{C}{(1 + t^2)^2} + \frac{\tan^{-1} t}{(1 + t^2)^2} \end{aligned}$$

The first term goes to 0 for any initial condition. Since $|\tan^{-1} t| \leq \pi/2$ the second term also goes to 0.

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Integrating Factor Summary

$$y' + ay = 0$$

\Downarrow

$$y' + ay = g(t)$$

\Downarrow

$$y' + p(t)y = g(t)$$

\Downarrow

$$\text{general solution: } y(t) = \frac{1}{\mu(t)} \left[C + \int \mu(t)g(t) dt \right], \quad \mu(t) = \int p(t) dt$$

$$\text{IVP solution: } y(t) = \frac{1}{\mu(t)} \left[y_0 + \int_{t_0}^t \mu(s)g(s) ds \right], \quad \mu(t_0) = 1$$

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