Set 4: First Order ODEs - Part 3

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Nonlinear First Order ODEs

Suppose we have a first order nonlinear ODE:

$$\frac{dy}{dx} = f(x, y)$$

Note the change in notation and that we can always write this as:

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0$$

Differential Form:

$$M(x, y)dx + N(x, y)dy = 0$$

Nonlinear First Order ODEs

- We can treat x as the independent variable, i.e., $y = \phi(x)$
- We can treat y as the independent variable, i.e., $x = \theta(y)$
- What are the integral curves?
- When is an ODE defined for a choice of independent variable?
- Can we find $y = \phi(x)$ or $x = \theta(y)$?

We consider a class of nonlinear first order ODEs that can be solved by direct integration.

Separable Nonlinear First Order ODEs

Definition 4.1. An ODE is separable if it can be written as

$$M(x) + N(y)\frac{dy}{dx} = 0$$

Note that it is still nonlinear since there is a product of dy/dx with y (assuming $N(y)$ is nontrivial).

Separable Example

$$\frac{dy}{dx} = -\frac{x}{y} \tag{1}$$

$$y\frac{dy}{dx} = -x \tag{2}$$

$$x + y\frac{dy}{dx} = 0 \tag{3}$$

$$M(x) = x \quad \text{and} \quad N(y) = y \tag{4}$$

Solution via Direct Integration

Recall the Chain Rule:

$$\frac{d}{dx}G(y(x)) = \frac{dG}{dy}\frac{dy}{dx}$$

$$\therefore \quad \int \Big[\frac{d}{dx}G(y(x))\Big]dx = \int \Big[\frac{dG}{dy}\frac{dy}{dx}\Big]dx$$

$$= G(y(x)) + C$$

Solution via Direct Integration

Applying to ODE (1):

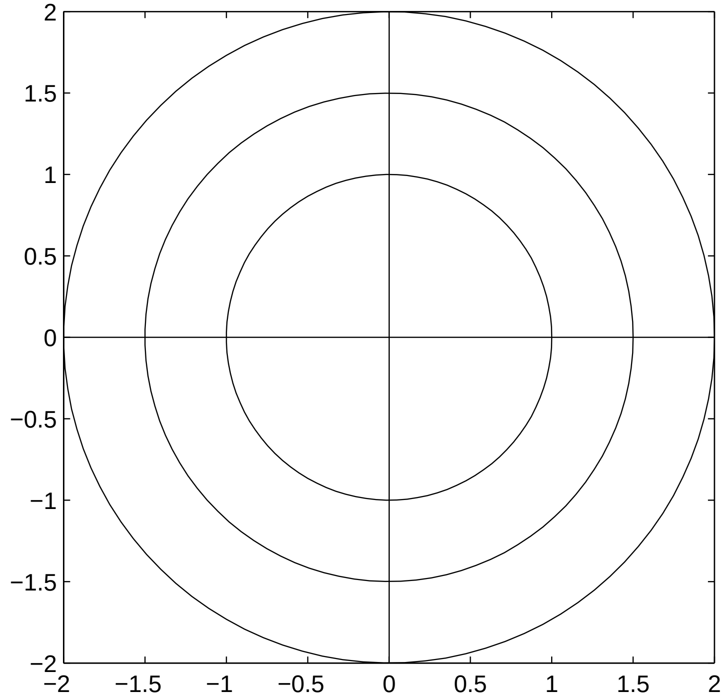
$$y\frac{dy}{dx} = -x$$

$$\int [y\frac{dy}{dx}]\,dx = \int y\ dy = -\int x\ dx$$

$$\frac{1}{2}y^2 = -\frac{1}{2}x^2 + C$$

$$x^2 + y^2 = C$$

Example: Integral Curves



Integral curves for $y' = -x/y$

Solution via Direct Integration

The ODEs

$$\frac{dy}{dx} = -\frac{x}{y} \tag{5}$$

$$\frac{dx}{dy} = -\frac{y}{x} \tag{6}$$

have integral curves

$$x^2 + y^2 = C, \quad C > 0 \tag{7}$$

- Implicit form
- Any differentiable function, $y = \phi(x)$, that satisfies (7) solves (5)
- Any differentiable function, $x = \theta(y)$, that satisfies (7) solves (6)

Direct Integration of a Separable ODE

Is there a method of producing integral curves?

Recall:

$$y\frac{dy}{dx} = -x$$

$$x + y\frac{dy}{dx} = M(x) + N(y)\frac{dy}{dx} = 0$$

$$\frac{1}{2}y^2 = \int [y\frac{dy}{dx}]\,dx = \int y\;dy = \int N(y)\;dy$$

$$-\frac{1}{2}x^2 = -\int x\;dx = -\int M(x)\;dx$$

$$\Downarrow$$

$$x^2 + y^2 = C$$

Direct Integration of a Separable ODE

Let $H_1(x)$ and $H_2(y)$ be the antiderivatives of $M(x)$ and $N(y)$ respectively.

$$\frac{d}{dx}H_1(x) = M(x) \quad \text{and} \int M(x)dx = H_1(x) + C$$

$$\frac{d}{dy}H_2(y) = N(y) \quad \text{and} \int N(y)dy = H_2(y) + C$$

Direct Integration of a Separable ODE

$$H_1(x) + H_2(y) = C$$

$$\frac{d}{dx}\big[H_1(x) + H_2(y)\big] = \frac{dC}{dx}$$

$$\frac{dH_1}{dx}(x) + \frac{dH_2}{dy}(y)\frac{dy}{dx} = 0$$

$$M(x) + N(y)\frac{dy}{dx} = 0$$

So the integral curves of a separable first order ODE are given by

$$H_1(x) + H_2(y) = C$$

where, $\int M(x)dx = H_1(x)$ and $\int N(y)dy = H_2(y)$

Explict Forms and IVPs

- Implicit form: $y = \phi(x)$?.
- dy/dx not defined when $y = 0$

$$x + y\frac{dy}{dx} = 0 \quad \text{and} \quad x^2 + y^2 = C$$

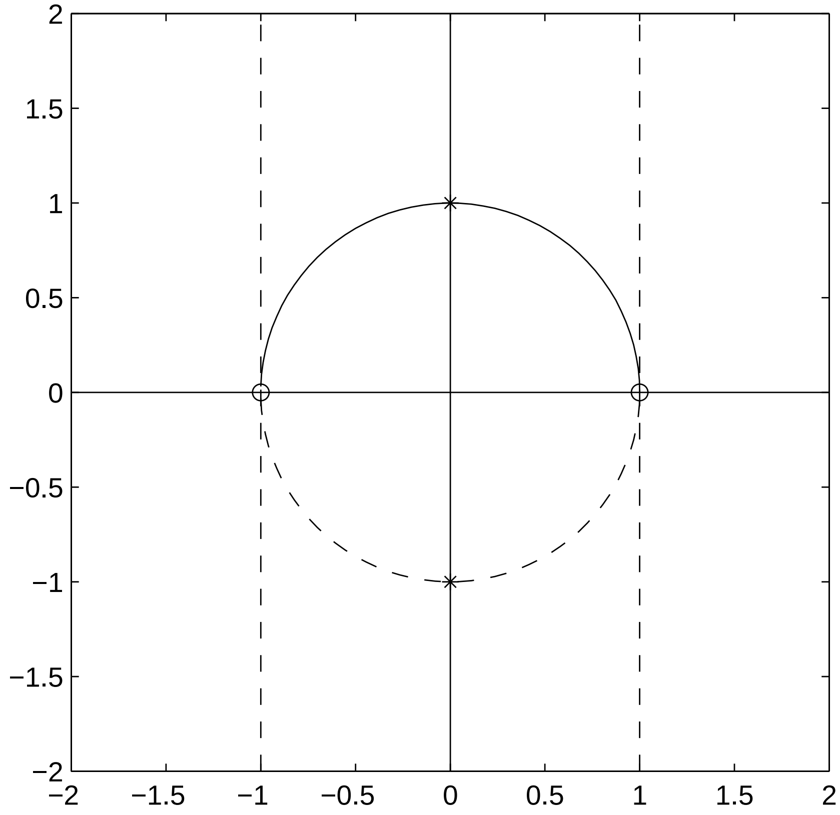
$$y = \pm\sqrt{C - x^2}$$

$$y(0) = 1 \rightarrow y = \sqrt{1 - x^2} \quad \text{and}$$

$$y(0) = -1 \rightarrow y = -\sqrt{1 - x^2}$$

- solutions defined only for $-1 \leq x \leq 1$
- Each IVP has unique solution only for $-1 < x < 1$ even though dy/dx is defined everywhere except on x-axis, i.e., $y = 0$

Example: Integral Curves and IVPs



Integral curves and two IVPs for $y' = -x/y$

Explict Forms and IVPs

- Two solutions on $-1 < x < 1$ satisfy $y(-1) = 0$.
- Two solutions on $-1 < x < 1$ satisfy $y(1) = 0$.
- Similar statements for $x = \theta(y)$.
- Cannot always find $y = \phi(x)$ or $x = \theta(y)$.
- Nonlinear existence and uniqueness is complicated!
- Entire integral curve $x^2 + y^2 = 1$ can be generated by using $x(t)$ and $y(t)$ and

$$\frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)} = -\frac{x}{y} \longrightarrow \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Example

$$\frac{dy}{dx} = \frac{4x - x^3}{4 + y^3}$$

$$M(x)dx + N(y)dy = (x^3 - 4x)dx + (y^3 + 4)dy = 0$$

$$H_1(x) = \int (x^3 - 4x)dx = \frac{1}{4}x^4 - 2x^2$$

$$H_2(y) = \int (y^3 + 4)dy = \frac{1}{4}y^4 + 4y$$

$$y^4 + 16y + x^4 - 8x^2 = C$$

Example

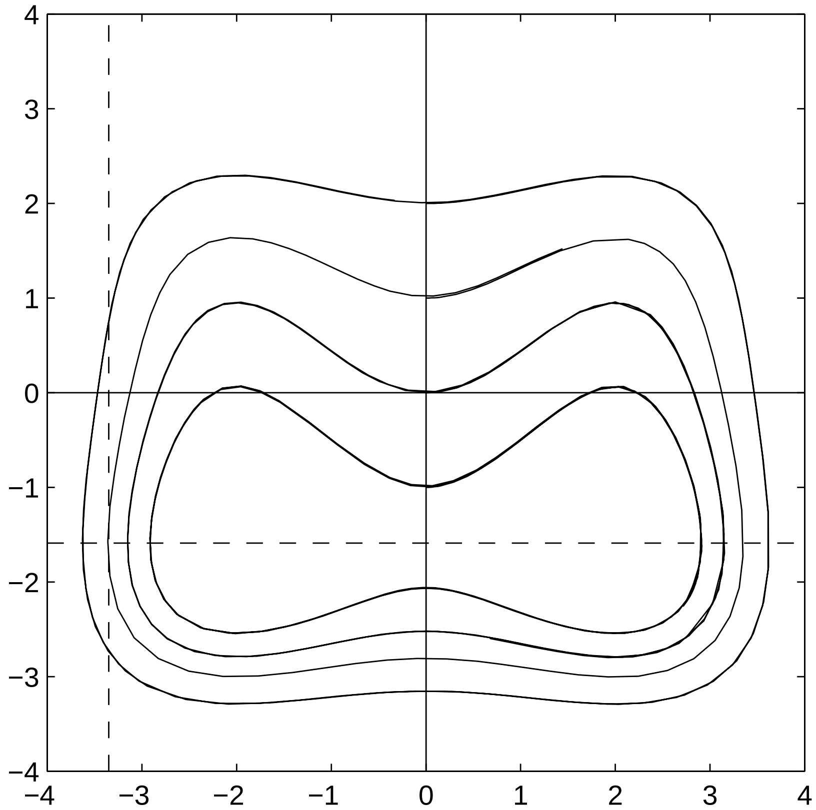
An IVP:

$$\frac{dy}{dx} = \frac{4x - x^3}{4 + y^3}, \quad (x_0, y_0) = (0, 1)$$

$$y^4 + 16y + x^4 - 8x^2 = 17$$

$$y = (-4)^{1/3} \approx -1.59 \rightarrow \frac{dy}{dx} = \infty$$

Example: Integral Curves and IVPs



Integral curves for $\frac{dy}{dx} = \frac{4x - x^3}{4 + y^3}$