Set 5: First Order ODEs - Part 4

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Ordinary Differential Equations
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Linear First Order ODEs

Theorem 5.1 (Textbook page 68)**.** *If* $p(t)$ *and* $g(t)$ *are continuous on an open interval* $\mathcal{I} : \alpha < t < \beta$ *then there exists a unique solution* $y = \phi(t)$ *on* $\mathcal{I}$ *to the initial value problem*

$$y' + p(t)y = g(t)$$

$$y(t_0) = y_0, \quad \alpha < t_0 < \beta$$

for any value of y_0.

Linear First Order ODEs

Corollary 5.2. *The unique solution* $y = \phi(t)$ *on* $\mathcal{I}$ *to the initial value problem under the conditions of Theorem 5.1 is given by*

$$y(t) = \frac{1}{\mu(t)}\left[y_0 + \int_{t_0}^{t} \mu(s)g(s)ds\right]$$

$$\mu(t) = e^{z(t)}$$

$$z(t) = \int_{t_0}^{t} p(t)dt$$

(Note that $\mu(t_0) = 1$.*)*

Linear First Order ODEs

Corollary 5.3. *All possible solutions to the differential equation on $\mathcal{I}$ under the conditions of Theorem 5.1 are given by the general solution*

$$y(t) = \frac{1}{\mu(t)}\left[C + \int_{t_0}^{t} \mu(s)g(s)ds\right]$$

$$\mu(t) = e^{z(t)}$$

$$z(t) = \int_{t_0}^{t} p(t)dt$$

Linear First Order ODEs

Assuming the conditions of Theorem 5.1:

- A general solution in explicit form is known that characterizes all solutions to the ODE.
- A particular solution to an IVP results by setting $C = y_0$
- The solution requires only two antiderivatives.
- Possible points of discontinuity or singularity of the solution $y(t)$ can be identifed from $p(t)$ and $g(t)$.
- The conditions given are sufficient not necessary. It is possible for the solution $y(t)$ to be continuous even when $p(t)$ and/or $g(t)$ are/is discontinuous.

Example

Recall, the linear first order ODE:

$$ty' + 2y = 4t^2 \to y' + \frac{2}{t}y = 4t$$

$$p(t) = \frac{2}{t} \quad \text{and} \quad g(t) = 4t$$

General solution: $y(t) = t^2 + \dfrac{C}{t^2}$

Possible problem points?

- $g(t)$ is continuous for $-\infty < t < \infty$, $\therefore$ no problem
- $p(t)$ is continuous for $t < 0$ and $t > 0$, $\therefore$ possible problem at $t = 0$

Some Solutions of Interest

Applying Theorem 5.1 to $y' + (2/t)y = 4t$ yields:

$$y(1) = 2 \rightarrow y(t) = t^2 + \frac{1}{t^2}, \quad 0 < t < \infty, \quad \mathit{discontinuous}$$

$$y(-1) = 2 \rightarrow y(t) = t^2 + \frac{1}{t^2}, \quad -\infty < t < 0, \quad \mathit{discontinuous}$$

$$y(1) = -1 \rightarrow y(t) = t^2 - \frac{2}{t^2}, \quad 0 < t < \infty, \quad \mathit{discontinuous}$$

$$y(-1) = -1 \rightarrow y(t) = t^2 - \frac{2}{t^2}, \quad -\infty < t < 0, \quad \mathit{discontinuous}$$

$$\therefore \text{discontinuous} \;\; y(t) \rightarrow \text{discontinuous} \;\; p(t)$$

Some Solutions of Interest

Applying Theorem 5.1 to $y' + (2/t)y = 4t$ yields:

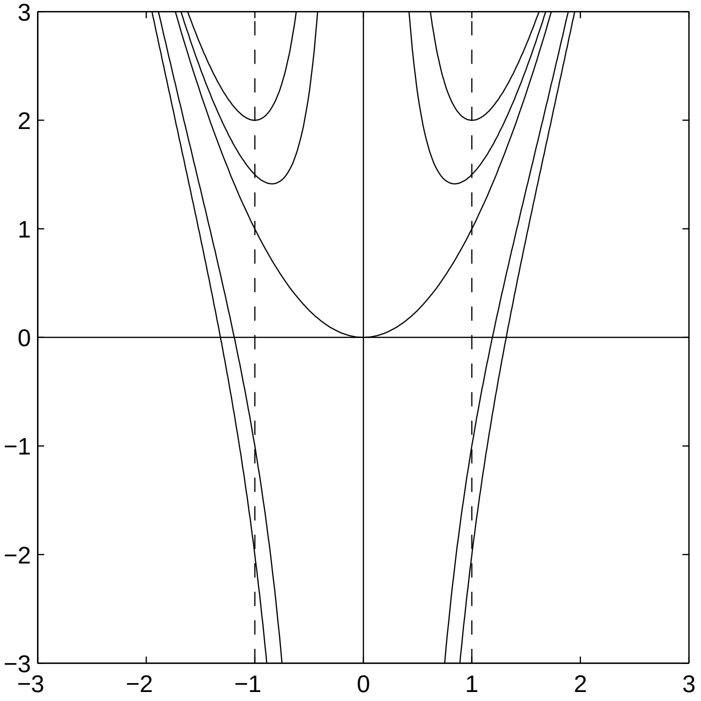
$$y(0) = 0 \rightarrow y(t) = t^2, \quad \infty < t < \infty$$

continuous on entire real line

$$\therefore \text{ discontinuous } \; p(t) \not\rightarrow \text{ discontinuous } \; y(t)$$

Discontinuous points of $p(t)$ and $g(t)$ only **possibly** a problem.

Example: Integral Curves



Nine solutions for $y' + \frac{2}{t}y = 4t$ with multiple initial conditions at

$$x = \pm 1, \ \ 0.$$

Nonlinear First Order ODEs

Theorem 5.4 (Textbook page 70)**.** *If $f(t, y)$ and its partial derivative $\partial f/\partial y$ are continuous in the rectangle*

$$\mathcal{R} = \left\{(t, y) : \alpha < t < \beta, \;\; \gamma < y < \delta\right\}$$

then there is a unique solution $y = \phi(t)$ of the initial value problem

$$y' = f(t, y), \quad y(t_0) = y_0, \quad (t_0, y_0) \in \mathcal{R}$$

defined on some subinterval around t_0

$$t_0 - h < t < t_0 + h$$

Nonlinear First Order ODEs

- $\partial f/\partial y$ is used because we want t to be the independent variable.
- The conditions are sufficient not necessary.
- The solution is guaranteed to exist on a subinterval not on the entire t interval defining $\mathcal{R}$. The subinterval is, in general, not easy to determine from the differential equation only.
- If f or $\partial f/\partial y$ are discontinuous somewhere in $\mathcal{R}$ the theorem says nothing about the situation. There may be none, one or more solutions on all or part of $\mathcal{R}$.

Nonlinear First Order ODEs

- There is no general solution form for an arbitrary nonlinear first order ODE or an associated IVP.
- Identifying the form of all possible solutions or the solution with $y(t_0) = y_0$ depends strongly on the class of nonlinear first order ODEs.
- We have seen one such class. For separable first order equations we have

$$M(x)dx + N(y)dy = 0$$

$$\int M(x)dx + \int N(y)dy = H_1(x) + H_2(y) = C$$

$$\int_{x_0}^{x} M(x)dx + \int_{y_0}^{y} N(y)dy = 0$$

Example

$$y' = \frac{(3x^2 + 4x + 2)}{(2y - 2)}$$

$y = 1 \rightarrow y' = \infty$ line of interest

$$f(x, y) = (3x^2 + 4x + 2)/(2y - 2)$$

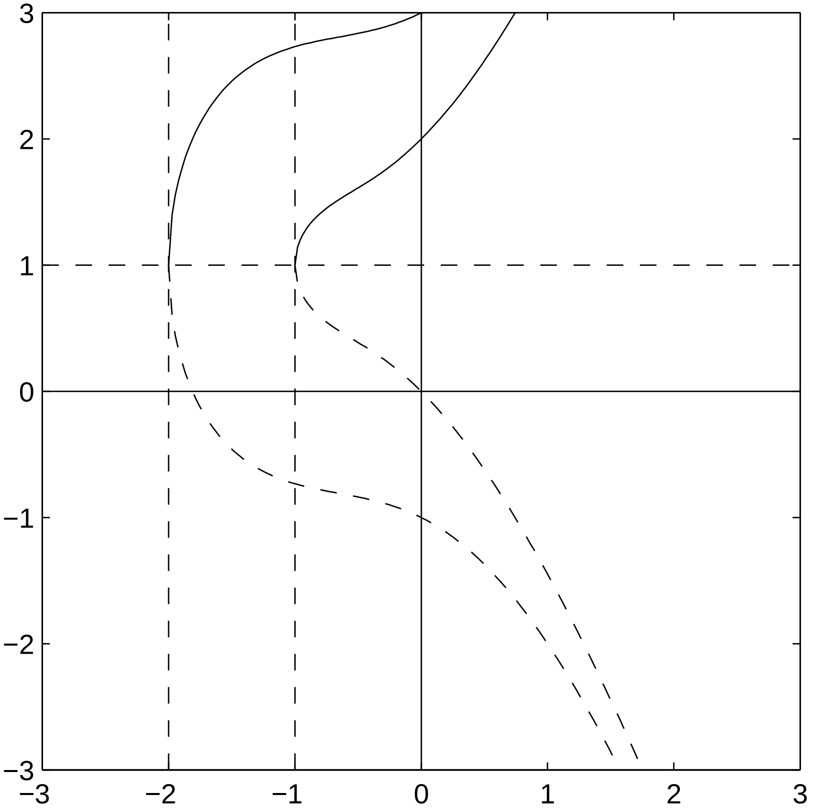
$$\frac{\partial f}{\partial y}(x, y) = -\frac{(3x^2 + 4x + 2)}{2(y - 1)^2}$$

$$y(x) = 1 \pm \sqrt{c + 1 + x^3 + 2x^2 + 2x}$$

Example

- Theorem 5.4 guarantees unique solution on some interval in x within any rectangle $\mathcal{R}$ that does not contain any part of the line $y = 1$.
- Consider initial conditions $(0, 3)$, $(0, 2)$, $(0, 0)$, and $(0, -1)$. All unique but not defined on entire x axis.
- Subinterval determined by where solution for a particular initial condition crosses $y = 1$.
- In this case the solutions become complex, i.e., they do not blow up in magnitude.
- Consider $(-2, 1)$ and $(-1, 1)$. Theorem 5.4 says nothing but two solutions each!

Example



$y' = (3x^2 + 4x + 2)/(2y - 2)$, $(x_0, y_0) = (0, 3),\ (0, 2),\ (0, 0),\ (0, -1)$.

Example

Some times the singularities, i.e., points where solutions go to ∞ in magnitude, are not expected from the ODE form.

$$y' = y^2, \quad y(0) = y_0$$

$$f(t, y) = y^2$$

$$\frac{\partial f}{\partial y}(t, y) = 2y$$

Continuous everywhere in (t, y) plane therefore a unique solution exists for any initial condition

Example

$$y(t_0) = y_0 \to y(t) = \frac{y_0}{1 - y_0 t}$$

$$\lim_{t \to \frac{1}{y_0}} |y(t)| = \infty$$

$$\text{if } y_0 > 0 \text{ then } \quad |y(t)| < \infty, \quad -\infty < t < \frac{1}{y_0}$$

$$\text{if } y_0 < 0 \text{ then } \quad |y(t)| < \infty, \quad \frac{1}{y_0} < t < \infty$$

- There is a singularity at a value of t that depends on the initial condition y_0.
- The ODE gives no indication that there is a problem!

Discontinuous Coefficients

Consider the linear first order ODE

$$y' + p(t)y = g(t), \quad y(t_0) = y_0$$

Suppose there is a jump discontinuity in $p(t)$ and/or $g(t)$ at some point t_d.

The problem can be solved by solving the problems:

$$\begin{aligned} y_1' + p(t)y_1 &= g(t), \quad y_1(t_0) = y_0, \quad t_0 \le t \le t_d \\ y_2' + p(t)y_2 &= g(t), \quad y_2(t_d) = y_1(t_d), \quad t_d < t \end{aligned}$$

This is done by exploiting the fact that we have two constants at our disposal from the two general solutions.

Example

$$y' + 2y = g(t), \quad y(0) = 0$$

$$g(t) = \begin{cases} 1 & \text{if } 0 \leq t \leq 1 \\ 0 & \text{if } t > 1 \end{cases}$$

$$y_1' + 2y_1 = g(t), \quad y_1(0) = 0, \quad 0 \leq t \leq 1$$

$$\mu(t) = e^{2t}$$

$$y_1(t) = C_1 e^{-2t} + e^{-2t} \int e^{2t} dt = C_1 e^{-2t} + \frac{1}{2}$$

$$y_1(0) = 0 \rightarrow C_1 = -\frac{1}{2}, \quad y_1(t) = \frac{1}{2}\left(1 - e^{-2t}\right)$$

Example

$$y_2' + 2y_2 = g(t) = 0, \quad t > 1, \ \ y_2(1) = y_1(1)$$

$$y_2(t) = C_2 e^{-2t}$$

$$y_2(1) = C_2 e^{-2} = y_1(1) = \frac{1}{2}\big(1 - e^{-2}\big)$$

$$\therefore C_2 = \frac{1}{2}\big(e^2 - 1\big) \quad \text{and} \quad y_2(t) = \frac{1}{2}\big(e^2 - 1\big)e^{-2t}$$

$$y(t) = \begin{cases} \frac{1}{2}\big(1 - e^{-2t}\big) & \text{if } 0 \le t \le 1 \\ \frac{1}{2}\big(e^2 - 1\big)e^{-2t} & \text{if } t > 1 \end{cases}$$