

# **Set 6: First Order ODEs - Part 5**

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## First Order ODEs

Recall the chain rule and its relationship to the ODE with solution  $y = \phi(x)$  and its integral curves:

$$\Psi(x, \phi(x)) = \Psi(x, y) = C$$

$$\frac{d}{dx} \Psi(x, y) = \frac{\partial \Psi}{\partial x}(x, y) + \frac{\partial \Psi}{\partial y}(x, y) \frac{dy}{dx} = 0$$

$$\frac{\partial \Psi}{\partial x}(x, y) = \Psi_x(x, y) = M(x, y)$$

$$\frac{\partial \Psi}{\partial y}(x, y) = \Psi_y(x, y) = N(x, y)$$

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0$$

## First Order ODEs

So given  $\Psi(x, y) = C$  we can always create an ODE

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0$$

such that

$$\frac{\partial \Psi}{\partial x}(x, y) = \Psi_x(x, y) = M(x, y)$$

$$\frac{\partial \Psi}{\partial y}(x, y) = \Psi_y(x, y) = N(x, y)$$

$$[\Psi(x, y)]' = M(x, y) + N(x, y) \frac{dy}{dx}$$

$$\Psi(x, y) = \int [M(x, y) + N(x, y) \frac{dy}{dx}] dx = C$$

Direct integration works.

## First Order ODEs

What about the other direction?

Given an ODE

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0$$

does there always exist  $\Psi(x, y)$  such that

$$\Psi_x(x, y) = M(x, y) \quad \text{and} \quad \Psi_y(x, y) = N(x, y)?$$

In general, no.

Assume  $\Psi(x, y)$  exists and determine conditions that must be satisfied.

## First Order ODEs

If  $\Psi_x(x, y) = M(x, y)$  we can integrate with respect to  $x$  to get a form of  $\Psi$ :

$$\Psi(x, y) = \int \Psi_x(x, y) dx = \int M(x, y) dx$$

$$\int M(x, y) dx = Q(x, y) + C \quad \text{not general enough}$$

$$\int M(x, y) dx = Q(x, y) + h(y)$$

$h(y)$  differentiable, arbitrary function plays the role of a constant w/r to  $x$ .

## First Order ODEs

Then, if  $\Psi_y(x, y) = N(x, y)$  we can differentiate with respect to  $y$

$$\Psi(x, y) = Q(x, y) + h(y)$$

$$\Psi_y(x, y) = Q_y(x, y) + \frac{d}{dy}h(y)$$

$$N(x, y) = Q_y(x, y) + \frac{d}{dy}h(y) \rightarrow h'(y) = N(x, y) - Q_y(x, y)$$

Solve ODE for  $h(y)$  and we are done?

No! We need constraints on ODE.

## First Order ODEs

Solving the ODE

$$h'(y) = N(x, y) - Q_y(x, y)$$

completely determines  $h(y)$  only if there is no dependence on  $x$ .

So we need the following condition:

$$\frac{\partial}{\partial x} [N(x, y) - Q_y(x, y)] = 0$$

## First Order ODEs

Relate the condition to the ODE:

$$\text{If } \frac{\partial}{\partial x} [N(x, y) - Q_y(x, y)] = 0$$

$$\frac{\partial N}{\partial x}(x, y) - \frac{\partial}{\partial x} Q_y(x, y) = 0$$

$$\frac{\partial N}{\partial x}(x, y) - \frac{\partial}{\partial y} Q_x(x, y) = 0$$

$$\Psi(x, y) = Q(x, y) + h(y) \rightarrow \Psi_x(x, y) = Q_x(x, y) = M(x, y)$$

$$\therefore \frac{\partial N}{\partial x}(x, y) - \frac{\partial M}{\partial y}(x, y) = 0$$

## First Order ODEs

So if  $N_x(x, y) = M_y(x, y)$  then we can determine  $h(y)$  by solving

$$h'(y) = N(x, y) - Q_y(x, y)$$

Given  $h(y)$  we have:

$$\Psi(x, y) = Q(x, y) + h(y)$$

$\Psi(x, y) = C$  defines integral curves for

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0$$

$\Psi(x, y) = C$  defines an implicit relationship between  $x$  and  $y$ .

## Example

$$(y \cos x + 2xe^y) + (\sin x + x^2e^y - 1)y' = 0$$

$$M_y = \cos x + 2xe^y$$

$$N_x = \cos x + 2xe^y$$

$\therefore$  The system of equations

$$\Psi_x = y \cos x + 2xe^y$$

$$\Psi_y = \sin x + x^2e^y - 1$$

has a solution  $\Psi(x, y)$

## Example

$$\Psi_x = M = y \cos x + 2xe^y \quad (1)$$

$$\Psi_y = N = \sin x + x^2 e^y - 1 \quad (2)$$

Integrate (1)

$$\begin{aligned} \Psi &= \int \Psi_x dx = \int M dx = h(y) + \int (y \cos x + 2xe^y) dx \\ &= h(y) + y \sin x + x^2 e^y = h(y) + Q \end{aligned}$$

Differentiate  $\Psi$  and use (2)

$$N = \Psi_y = Q_y + h'$$

$$\sin x + x^2 e^y - 1 = h' + \sin x + x^2 e^y$$

$$h' = -1 \rightarrow h(y) = -y$$

## Example

$$(y \cos x + 2xe^y) + (\sin x + x^2e^y - 1)y' = 0$$

$$\Psi(x, y) = Q(x, y) + h(y)$$

$$Q(x, y) = y \sin x + x^2e^y$$

$$h(y) = -y$$

$$\therefore \Psi(x, y) = y \sin x - y + x^2e^y = C$$

## Example

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0$$

$$(2xy^2 + 2y) + (2x^2y + 2x)y' = 0$$

$$M_y = 4xy + 2$$

$$N_x = 4xy + 2$$

exact differential equation

Therefore we have the system of equations

$$\Psi_x(x, y) = M(x, y) \tag{3}$$

$$\Psi_y(x, y) = N(x, y) \tag{4}$$

## Example

Use (3) to get  $Q(x, y)$

$$\Psi = \int M dx = \int (2xy^2 + 2y) dx = Q + h(y) = (x^2y^2 + 2xy) + h(y)$$

Use (4) to get  $h(y)$

$$Q_y + h' = (2x^2y + 2x)$$

$$h' = (2x^2y + 2x) - (2x^2y + 2x) = 0$$

$$\Psi(x, y) = x^2y^2 + 2xy = C$$

## Example

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0$$

$$(3xy + y^2) + (x^2 + xy) \frac{dy}{dx} = 0$$

$$M_y = \frac{\partial}{\partial y}(3xy + y^2) = 3x + 2y$$

$$N_x = \frac{\partial}{\partial x}(x^2 + xy) = 2x + y$$

not an exact differential equation

## Exact Equations

**Definition 6.1.** The first order ODE

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0$$

is an exact differential equation on an open rectangle  $\alpha < x < \beta$ ,  
 $\gamma < y < \delta$  if and only if on the rectangle

$$M_y(x, y) = N_x(x, y)$$

## Exact Equations

**Theorem 6.1.** *If  $M(x, y)$ ,  $N(x, y)$ ,  $M_y(x, y)$ , and  $N_x(x, y)$  are continuous on an open rectangle  $\alpha < x < \beta$ ,  $\gamma < y < \delta$  then there exists a function  $\Psi(x, y)$  such that*

$$\Psi_x(x, y) = M(x, y) \quad \text{and} \quad \Psi_y(x, y) = N(x, y)$$

*if and only if*

$$M_y(x, y) = N_x(x, y)$$

## Example

Textbook p. 99 Problem 1

$$M(x, y) + N(x, y)y' = 0$$

$$(2x + 3) + (2y - 2)y' = 0$$

$$M_y = 0$$

$$N_x = 0$$

exact and separable

$$H_1(x) = \int (2x + 3)dx = x^2 + 3x$$

$$H_2(y) = \int (2y - 2)dy = y^2 - 2y$$

$$x^2 + 3x + y^2 - 2y = C$$

## Example

Textbook p. 99 Problem 2

$$M(x, y) + N(x, y)y' = 0$$

$$(2x + 4y) + (2x - 2y)y' = 0$$

$$M_y = 4$$

$$N_x = 2$$

not exact

## Example

Textbook p. 99 Problem 3

$$M(x, y) + N(x, y)y' = 0$$

$$(3x^2 - 2xy + 2) + (6y^2 - x^2 + 3)y' = 0$$

$$M_y = -2x$$

$$N_x = -2x$$

$$\Psi = \int M(x, y)dx = Q(x, y) + h(y)$$

$$= h(y) + \int (3x^2 - 2xy + 2)dx = h(y) + (x^3 - x^2y + 2x)$$

## Example

Textbook p. 99 Problem 3

$$Q(x, y) = (x^3 - x^2y + 2x) \quad \text{and} \quad N(x, y) = Q_y + h'$$

$$(6y^2 - x^2 + 3) = (-x^2) + h'$$

$$\frac{dh}{dy} = 6y^2 + 3$$

$$h(y) = 2y^3 + 3y$$

$$\Psi(x, y) = Q(x, y) + h(y) = x^3 - x^2y + 2x + 2y^3 + 3y$$

## Summary

- General first order ODE for  $y = \phi(x)$

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0$$

- Separable differential equations  $M_y = N_x = 0$ .
- Exact differential equation if  $M_y = N_x$ .
- Exact differential equation if and only if  $\exists \Psi(x, y) = C$  such that

$$M(x, y) = \Psi_x(x, y) \quad N(x, y) = \Psi_y(x, y) \quad (5)$$

- Exact equations can be solved by solving system (5).