Set 7: First Order ODEs - Part 6

Kyle A. Gallivan

Department of Mathematics

Florida State University

Ordinary Differential Equations
Fall 2009

First Order ODEs

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

is exact if

$$M_y = N_x$$

What can be done if it is not exact?

Find a function $\mu(x,y)$ that makes

$$\mu(x,y)M(x,y) + \mu(x,y)N(x,y)\frac{dy}{dx} = 0$$

an exact differential equation.

Exact Equations and Integrating Factors

- $\mu(x, y)$ may not exist
- if $\mu(x, y)$ exists is it not unique
- In general, finding $\mu(x, y)$ is just as difficult as solving the ODE.

Exact Equations and Integrating Factors

$$M + Ny' = (3xy + y^2) + (x^2 + xy)y' = 0$$

$$M_y = 3x + 2y$$
$$N_x = 2x + y$$

$$\mu(x, y) = x$$
$$(\mu M)_y = \left[3x^2y + xy^2\right]_y = 3x^2 + 2xy$$
$$(\mu N)_x = \left[x^3 + x^2y\right]_x = 3x^2 + 2xy$$

Exact ODE

Integrating Factors

Find a function $\mu(x, y)$ that makes

$$\mu(x, y)M(x, y) + \mu(x, y)N(x, y)\frac{dy}{dx} = 0$$

an exact differential equation.

$$(\mu M)_y - (\mu N)_x = 0$$

$$M\mu_y + M_y\mu - N\mu_x - N_x\mu = 0$$

$$M\mu_y - N\mu_x + (M_y - N_x)\mu = 0$$

- In general, this PDE is just as difficult to solve as the ODE.
- We must assume additional constraints to get a computable μ.

Integrating Factor

Suppose we look for a $\mu(x, y)$ that is a function of x only:

$$\mu_y(x, y) = 0$$

$$\mu_x(x, y) = \frac{d\mu}{dx} = \mu'$$

$$M\mu_y - N\mu_x + (M_y - N_x)\mu = 0$$

$$(M_y - N_x)\mu - N\mu' = 0$$

Integrating Factor

We therefore have for $\mu(x)$

$$(M_y - N_x)\mu - N\mu' = 0$$

Not necessarily solvable!

Assume

$$\frac{(M_y - N_x)}{N}$$

is a function of x only.

Integrating Factor

We have

$$(M_y - N_x)\mu - N\mu' = 0$$

$$\mu' - \frac{(M_y - N_x)}{N}\mu = 0 = \mu' + p(x)\mu$$

linear ODE

$$\frac{(M_y - N_x)}{N} - \frac{1}{\mu}\mu' = 0 = \tilde{M}(x) + \tilde{N}(\mu)\mu'$$

separable ODE

Example

$$M + Ny' = (3xy + y^2) + (x^2 + xy)y' = 0$$

$$M = (3xy + y^2) \quad N = (x^2 + xy)$$

$$M_y = 3x + 2y \quad N_x = 2x + y$$

$$\frac{(M_y - N_x)}{N} = \frac{(3x + 2y) - (2x + y)}{(x^2 + xy)} = \frac{(x + y)}{x(x + y)} = \frac{1}{x}$$

Example

$$\mu' - \frac{(M_y - N_x)}{N}\mu = 0$$

$$\mu' - \frac{1}{x}\mu = 0$$

$$p(x) = -\frac{1}{x} \quad g(x) = 0$$

$$\mu(x) = x$$

Example

The integrating factor may also be a function of y only.

$$M + Ny' = \left[4\frac{x^3}{y^2} + \frac{3}{y}\right] + \left[3\frac{x}{y^2} + 4y\right]y' = 0$$

$$M_y = -8x^3y^{-3} - 3y^{-2}$$

$$N_x = 3y^{-2}$$

$$P = \frac{M_y - N_x}{N} = \frac{-8x^3y^{-3} - 6y^{-2}}{3xy^{-2} + 4y}$$

$$\tilde{P} = \frac{N_x - M_y}{M} = \frac{8x^3y^{-3} + 6y^{-2}}{4x^3y^{-2} + 3y^{-1}}$$

$$= \frac{2}{y}$$

Example

$\tilde{P}$ is a function of y only therefore an integrating factor $\mu(y)$ exists defined by

$$\mu' = \tilde{P}\mu = \frac{2}{y}\mu$$

We therefore have

$$\int \frac{2}{y} dy = 2\ln|y| + c$$

$$\mu = Ce^{2\ln|y|} = C|y|^2 = Cy^2$$

$$\mu = y^2$$

Example

Applying $\mu(y) = y^2$ yields

$$\mu M + \mu N y' = (4x^3 + 3y) + (3x + 4y^3)y' = 0$$

$$\int (4x^3 + 3y)dx = x^4 + 3xy + h$$

$$\mu N = Q_y + h'$$

$$(3x + 4y^3) = 3x + h'$$

$$h' = 4y^3 \rightarrow h = y^4$$

$$\Psi = x^4 + 3xy + y^4 = C$$

Example

$$M + Ny' = (3xy + y^2) + (x^2 + xy)y' = 0$$

$$\mu(x,y) = \frac{1}{xy(2x+y)} \quad \text{is also an integrating factor}$$

$$\mu_x = -\frac{4xy + y^2}{x^2y^2(2x+y)^2}$$

$$\mu_y = -\frac{2x^2 + 2xy}{x^2y^2(2x+y)^2}$$

PDE satisfied $\quad M\mu_y - N\mu_x + (M_y - N_x)\mu = 0$

It generates the same solution to the ODE. But the algebra is much more tedious.

Summary for First Order ODEs

- Linear: $y' + p(t)y = g(t)$
  - integrating factor
  - known general solution form
- Nonlinear separable: $M(x) + N(y)y' = 0$
  - direct integration
  - known general solution form
- Nonlinear exact: $M(x, y) + N(x, y)y' = 0$
  - $M_y = N_x$
  - solve system of two equations via direct integration
  - attempt to find simple integrating factor when not exact