Set 8: First Order ODEs - Part 7

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Ordinary Differential Equations
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Analysis, Modeling, and Design

- Analyze the behavior as a function of parameters, e.g., initial condition
  - given just the ODE
  - after solving the ODE
- Model physical phenomena
  - adjust ODE based on knowledge of behavior from analysis of typical forms
  - model physical laws and their combinations
  - match observations
- design a system whose behavior is governed by an ODE
  - match desired behavior
  - analyze robustness of behavior

Autonomous ODEs

Definition 8.1. A first order ODE is autonomous if it can be written:

$$y' = f(y)$$

i.e., $f_t(t, y) = 0$.

Essentially, t is an independent value such that the actual value does not matter. Only $t - t_0$ is of interest in general.

The analysis we will do is often called phase analysis.

Population Growth

Recall,

$$y' = ry, \quad y(0) = y_0$$
$$y(t) = y_0 e^{rt}$$

- $r > 0$ Unbounded exponential growth.
- $r < 0$ exponential decay or damping to 0.
- Simple model of population growth has $r > 0$.
- Not realistic, needs to move from growth to decay based on size of population due to finite resources.

Population Growth

$$y' = h(y)y = (r - ay)y = r(1 - \frac{y}{K})y$$

$$r > 0, \quad a > 0, \;\; K = \frac{r}{a}$$

- small $y \to h(y) \approx r$, $y \approx e^{rt}$
- as y increases $h(y)$ decreases and y grows more slowly.
- once y is large enough $h(y) < 0$ and y decreases

Logistic Equation

$$y' = r(1 - \frac{y}{K})y$$

$$r > 0, \quad K > 0$$

- $r > 0$ is intrinsic growth rate.
- $K > 0$ is the saturation level.
- Analyze the ODE, i.e., $f(y)$, and predict the behavior of $y(t)$ on $0 < t$.

Logistic Equation

Find equilibrium solutions:

$$y' = f(y) = r(1 - \frac{y}{K})y$$
$$y' = 0 \rightarrow y(t) = C$$

roots of $f(y)$ quadratic in y

$$y = 0$$
$$y = K$$

Logistic Equation

$$y' = f(y) = r(1 - \frac{y}{K})y \quad \text{opens down}$$

$$y' = 0 \rightarrow y(t) = C \quad \text{roots of quadratic} \quad f(y)$$

$$\max f(y) \quad f' = 0 \quad \text{at} \quad y = \frac{K}{2}$$

$$0 < y < K \quad f(y) > 0 \quad y \text{ increasing}$$

$$y > K \quad f(y) < 0 \quad y \text{ decreasing}$$

Note $y = 0$ is therefore an unstable equilibrium.

Logistic Equation

Concavity of $y(t)$?

$$y' = f(y) = r(1 - \frac{y}{K})y$$

$$y'' = \frac{d}{dt} f(y) = f' \frac{dy}{dt} = f' f$$

$f', \quad f$ same signs $\rightarrow y$ concave up

$f', \quad f$ opposite signs $\rightarrow y$ concave down

Logistic Equation

Concavity of $y(t)$.

- $y = 0$ flat
- $0 < y < K/2$, $f' > 0$, $f > 0 \rightarrow y$ concave up increasing
- $K/2 < y < K$, $f' < 0$, $f > 0 \rightarrow y$ concave down increasing
- $y = K$ flat
- $K < y$, $f' < 0$, $f < 0 \rightarrow y$ concave up decreasing

Logistic Equation

Solve the ODE:

$$y' = f(y) = r(1 - \frac{y}{K})y$$

$$[\frac{1}{(1 - K^{-1}y)y}]y' = r$$

$$[\frac{1}{y} + \frac{K^{-1}}{(1 - K^{-1}y)}]y' = r$$

$$\int \frac{1}{y}\, dy + \int \frac{K^{-1}}{(1 - K^{-1}y)}\, dy = rt + c$$

$$z = 1 - K^{-1}y, \quad dz = -K^{-1}dy$$

$$\ln|y| - \ln|1 - K^{-1}y| = rt + c$$

Logistic Equation

$$\ln|y| - \ln|1 - K^{-1}y| = rt + c$$

$$\frac{y}{1 - K^{-1}y} = \frac{Ky}{K - y} = Ce^{rt}, \quad 0 \le y \le K$$

$$\frac{Ky_0}{K - y_0} = C, \quad 0 \le y \le K$$

$$y = \frac{CKe^{rt}}{K + Ce^{rt}}, \quad 0 \le y \le K$$

Can be shown solution is the same for $y > K$.

Logistic Equation

$$y = \frac{CKe^{rt}}{K + Ce^{rt}}$$

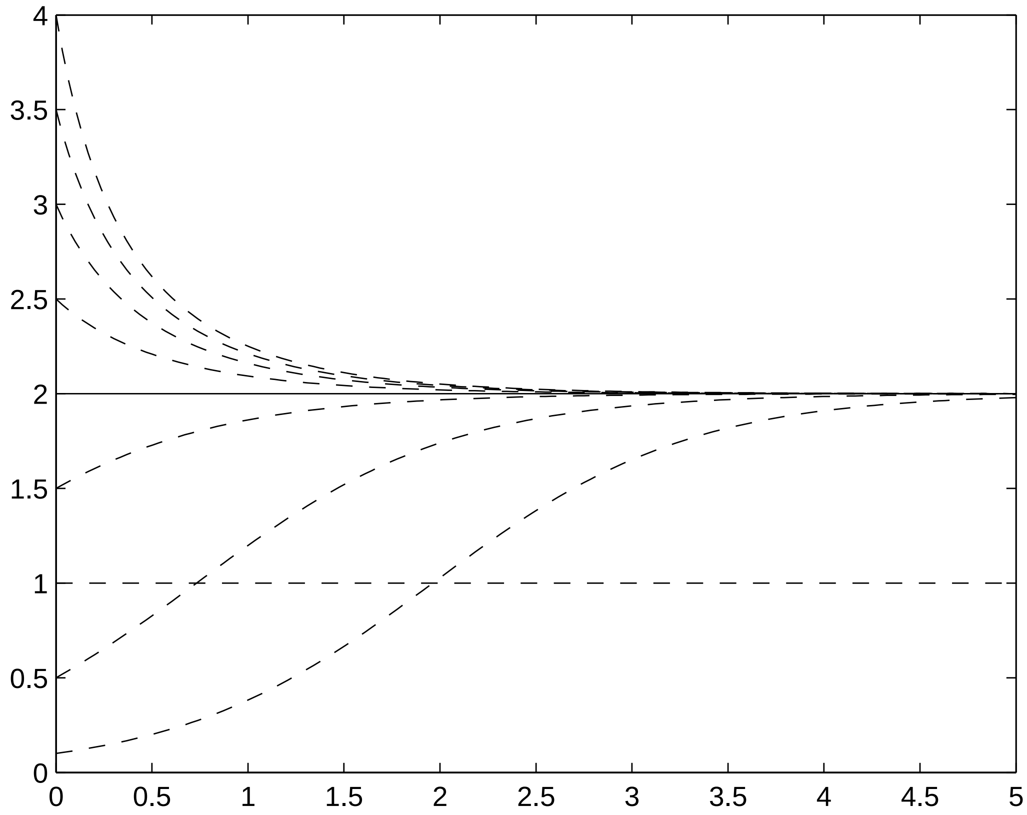
$$C = \frac{Ky_0}{K - y_0}$$

$$y(t) = \frac{Ky_0 e^{rt}}{K - y_0 + y_0 e^{rt}}$$

$$= \frac{Ky_0}{(K - y_0)e^{-rt} + y_0}$$

$$t \to \infty \quad y \to K$$

Logistic Equation



Logistic Equations $K = 2$, $r = 1.5$

Logistic Equation

- $y > K$ then y damps asymptotically to K
- $y = \epsilon > 0$ then y grows asymptotically to K
- physically the latter is a problem
- We would expect some interval around $0 < y < T$ where the population was not sustainable and would die out rather than grow to K
- Need another form of equation.

Threshold Equation

$$y' = f(y) = -r(1 - \frac{y}{T})y$$

$$r > 0, \quad T > 0$$

Analyze $f(y)$

$$f(y) = -r(1 - \frac{y}{T})y = \frac{r}{T}y^2 - ry$$

- Parabola opening up
- Roots at $y = 0$ and $y = T \rightarrow$ equilibrium solutions.
- $f(y) < 0$ for $0 < y < T$, y decreasing as t increases
- $f(y) > 0$ for $y > T$, y increasing as t increases

Threshold Equation

$f' = y''$?

- $f' = \frac{2r}{T}y - r \rightarrow f'(T/2) = 0$ minimum
- $f' < 0$ for $0 < y < T/2$
- $f' > 0$ for $T/2 < y < T$
- $f' > 0$ for $y > T$

Threshold Equation

Concavity.

- $0 < y < T/2$, $f' < 0$ and $f < 0 \rightarrow y$ concave up decreasing
- $T/2 < y < T$, $f' > 0$ and $f < 0 \rightarrow y$ concave down decreasing
- $y > T$, $f' > 0$ and $f > 0 \rightarrow y$ concave up increasing

Threshold Equation

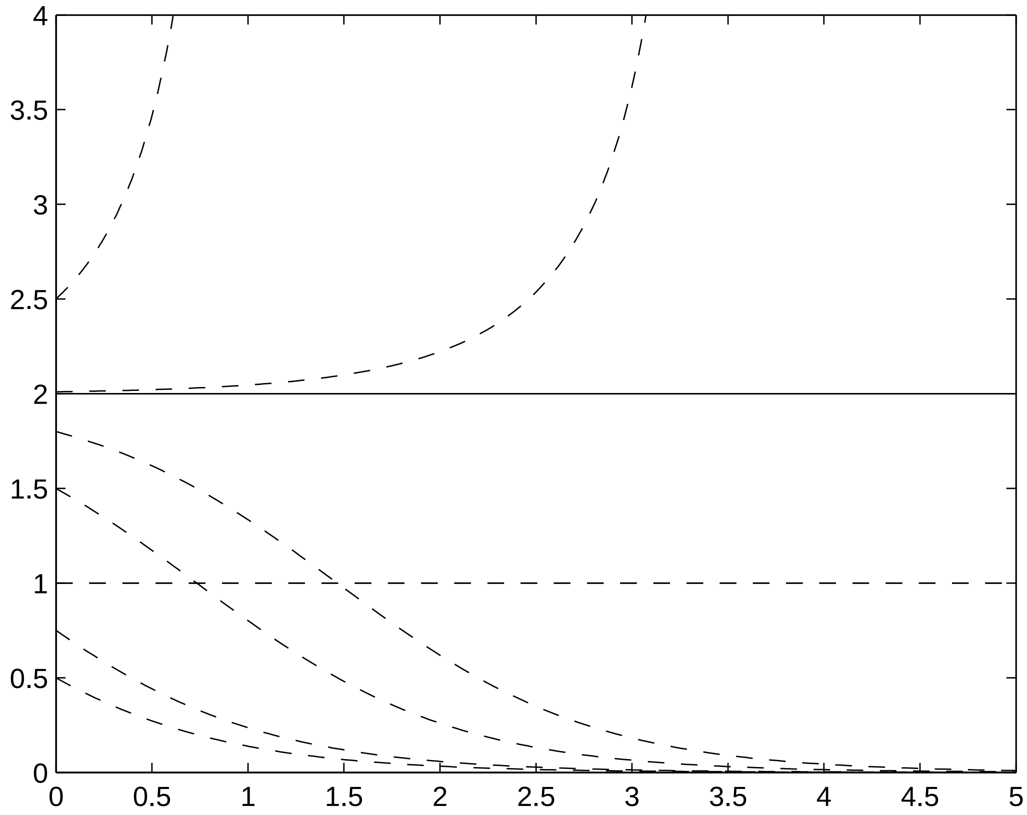
Solve the ODE

$$y' = f(y) = -r(1 - \frac{y}{T})y, \quad r > 0, \quad T > 0, \quad y(0) = y_0$$

$$y(t) = \frac{y_0 T}{y_0 + (T - y_0)e^{rt}}$$

- If $0 < y_0 < T$ then as $t \to \infty$ the solution $y \to 0$
- If $y_0 > T$ then as $t \to \infty$ the solution y has a vertical asymptote where $y_0 + (T - y_0)e^{rt} = 0$
- Asymptote at
$$t^* = \frac{1}{r} \ln \frac{y_0}{y_0 - T}$$

Threshold Equation



Threshold Equations $T = 2$, $r = 1.5$

Threshold Equation

- $0 < y < T$ the solution damps to 0
- $y > T$ exponential growth to a vertical asymptote
- Logistic equation gives reasonable behavior in forcing an upper bound (saturation) equilibrium.
- Logistic equation gives unreasonable behavior in only decaying to 0 when $y_0 = 0$.
- Threshold equation gives reasonable behavior in requiring a threshold before growth occurs. That is it damps when y is below the threshold.
- Threshold equation gives unreasonable behavior in resulting in unbounded growth above the threshold. (OK for some applications)
- Combine the equations.

Logistic Equation with Threshold

We encountered three equilibriums that we would like to maintain:

$$y = 0 < y = T < y = K$$

Want $y = 0$ and $y = K$ stable equilibriums

Want $y = T$ unstable equilibrium

$$y' = f(y) = -r(1 - \frac{y}{T})(1 - \frac{y}{K})y$$

Logistic Equation with Threshold

$$f(y) = -r(1 - \frac{y}{T})(1 - \frac{y}{K})y$$

$$0 < y < T \rightarrow f(y) < 0, \quad \therefore y \quad \text{decreasing}$$

$$T < y < K \rightarrow f(y) > 0, \quad \therefore y \quad \text{increasing}$$

$$K < y \rightarrow f(y) < 0, \quad \therefore y \quad \text{decreasing}$$

This is the desired general behavior.

Logistic Equation with Threshold

roots of quadratic f' are inflection points of y

$$y_1 = \frac{1}{3}\big(K + T + \sqrt{K^2 - KT + T^2}\big)$$

$$y_2 = \frac{1}{3}\big(K + T - \sqrt{K^2 - KT + T^2}\big)$$

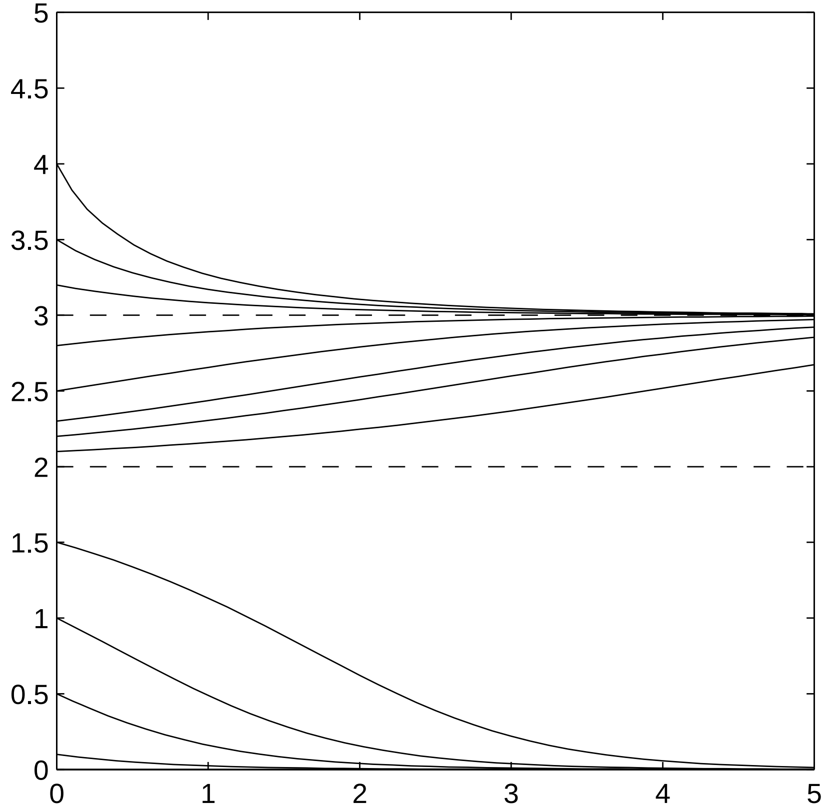
$$0 < y_2 < T < y_1 < K$$

Logistic Equation with Threshold

Concavity

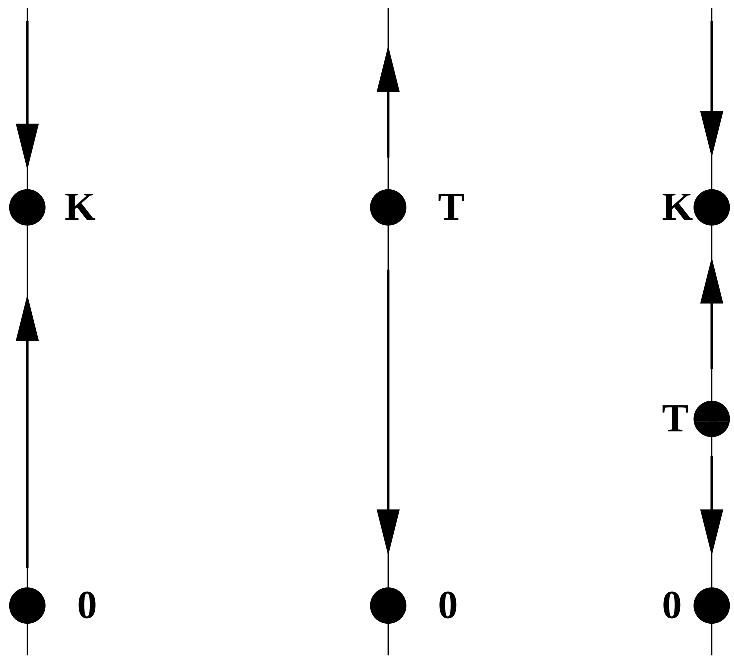
- $0 < y < y_2$, $f' < 0$, $f < 0$, $\therefore y$ concave up, decreasing
- $y_2 < y < T$, $f' > 0$, $f < 0$, $\therefore y$ concave down, decreasing
- $T < y < y_1$, $f' > 0$, $f > 0$, $\therefore y$ concave up, increasing
- $y_1 < y < K$, $f' < 0$, $f > 0$, $\therefore y$ concave down, increasing
- $K < y$, $f' < 0$, $f < 0$, $\therefore y$ concave up, decreasing

Logistic Equation with Threshold



Logistic with Threshold Equations $T = 2$, $K = 3$, $r = 1.5$

Phase Lines



Phase Lines for Logistic, Threshold, Combined ODEs