Set 9: Second Order Linear ODEs - Part 1

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Ordinary Differential Equations
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Second Order Linear ODEs

Definition 9.1. An ODE is second order and linear if it can be written

$$P(t)y'' + Q(t)y' + R(t)y = G(t)$$

An associated initial value problem also specifies

$$y(t_0) = y_0$$
$$y'(t_0) = y'_0$$

If $P(t) \neq 0$ the explict form

$$y'' + p(t)y' + q(t)y = g(t)$$

will often be used.

Second Order Linear ODEs

Definition 9.2. Given two functions p and q, the differential operator L associated with the ODE that maps a twice differentiable function of t on an interval $\alpha < t < \beta$ to another function of t is given by

$$L[\phi] = \phi'' + p\phi' + q\phi$$

Existence and Uniqueness

Theorem 9.1 (Textbook page 146). *If p,q, and g are continuous functions on the interval $\mathcal{I}$: $\alpha < t < \beta$ that contains t_0 then the initial value problem*

$$y'' + p(t)y' + q(t)y = g(t)$$
$$y(t_0) = y_0$$
$$y'(t_0) = y'_0$$

has a unique solution that exist throughout $\mathcal{I}$.

Superposition

Theorem 9.2 (Textbook page 147). *If y_1 and y_2 are two solutions to the homogeneous ODE*

$$L[y] = y'' + p(t)y' + q(t)y = 0$$

then the linear combination $c_1y_1 + c_2y_2$ is a solution for any constants c_1 and c_2.

Overview

- homogeneous second order linear ODEs with constant coefficients
- general solution and fundamental solution sets
- Wronskian
- general solution for nonhomogeneous problems
- Method of undetermined coefficients for nonhomogeneous constant coefficient problems
- Method of variation of parameters for nonhomogeneous constant coefficient problems
- some nonconstant coefficient problems

Constant Coefficient Problems

Definition 9.3. If a, b, , c are real constants, the ODE

$$ay'' + by' + cy = 0$$

is a second order linear ODE with constant coefficients.

- How are they solved?
- What are the important properties of the solutions?

Constant Coefficient Problems

Recall the first order problem

$$y' + cy = 0 \rightarrow y = e^{-ct}$$

Hypothesis for second order problem $y = e^{rt}$.

$$\begin{aligned} y &= e^{rt} \\ y' &= re^{rt} \\ y'' &= r^2e^{rt} \\ ay'' + by' + cy &= a(r^2e^{rt}) + b(re^{rt}) + c(e^{rt}) \\ &= [ar^2 + br + c]e^{rt} \end{aligned}$$

Constant Coefficient Problems

$$ay'' + by' + cy = \left[ar^2 + br + c\right]e^{rt}$$
$$e^{rt} \neq 0 \rightarrow ar^2 + br + c = 0$$

r generates a solution to the ODE if it is a root of $ar^2 + br + c$

Consistent with first order

$$y' + cy = 0$$
$$a = 0, \quad b = 1$$
$$br + c = 0 \leftrightarrow r = -c$$

Solutions

$$\text{ODE:} \quad ay'' + by' + cy = 0$$
$$\text{characteristic equation:} \quad ar^2 + br + c$$

$$b^2 - 4ac > 0 \rightarrow \text{two distinct real roots}$$
$$b^2 - 4ac < 0 \rightarrow \text{two complex conjugate roots}$$
$$b^2 - 4ac = 0 \rightarrow \text{repeated real root}$$

Distinct Real Roots

$$y'' - y = 0, \quad y(0) = 2, \quad y'(0) = -1$$
$$r^2 - 1 = 0 \rightarrow r = \pm 1$$
$$y_1 = c_1 e^t, \quad y_2 = c_2 e^{-t}$$
$$y_1(0) = 2 = c_1 e^0 = c_1$$
$$y_1'(0) = -1 \neq c_1 e^0 = 2 \times 1 = 2$$

Cannot satisfy the initial conditions with $y_1 y_1$ solves ODE but not the IVP.

Distinct Real Roots

$$y = c_1 e^t + c_2 e^{-t} \quad y' = c_1 e^t - c_2 e^{-t}$$
$$y(0) = c_1 + c_2 = 2$$
$$y'(0) = c_1 - c_2 = -1$$
$$c_1 = \frac{1}{2}, \quad c_2 = \frac{3}{2}$$

$$y = \frac{1}{2}e^t + \frac{3}{2}e^{-t}$$

To satisfy an arbitrary initial condition, we need two solutions.

Distinct Real Roots

- ODE: $ay'' + by' + cy$
- characteristic equation: $ar^2 + br + c$
- $b^2 - 4ac > 0$, two distinct real roots r_1 and r_2
- general solution: $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$
- any $y(t_0)$ and $y'(t_0)$ values can be satisfied by choosing c_1 and c_2

Distinct Real Roots

Three cases of asymptotic behavior:

- if both roots are negative then $y \to 0$
- if at least one root is positive then $y \to \pm\infty$
- if one root is 0 and the other is negative then $y \to C$

The initial condtion on y' determines the initial shape and the asymptotic behavior determines the final shape.

Example

$$y'' + 5y' + 6y = 0, \quad y(0) = 2, \quad y'(0) = 3$$
$$r^2 + 5r + 6 = (r+2)(r+3)$$
$$y = c_1 e^{-2t} + c_2 e^{-3t}$$
$$c_1 + c_2 = 2$$
$$-2c_1 - 3c_2 = 3$$
$$c_1 = 9, \quad c_2 = -7$$

$$y = 9e^{-2t} - 7e^{-3t}$$

Complex Conjugate Roots

$$ay'' + by' + cy = 0$$
$$ar^2 + br + c = 0$$
$$b^2 - 4ac < 0 \rightarrow \text{two complex conjugate roots}$$
$$r_1 = \lambda + i\mu, \quad r_2 = \lambda - i\mu$$
$$y_1 = e^{(\lambda + i\mu)t}, \quad y_2 = e^{(\lambda - i\mu)t}$$

Euler's Formula

Definition 9.4 (Euler's Formula). For any real μ

$$e^{i\mu t} = \cos \mu t + i \sin \mu t$$

We therefore have,

$$\begin{aligned} e^{(\lambda + i\mu)t} &= e^{\lambda t} e^{i\mu t} \\ &= e^{\lambda t} \cos \mu t + e^{\lambda t} i \sin \mu t \end{aligned}$$

Complex Conjugate Roots

We want a real solution → use linear combination.

$$\begin{aligned} y_1 &= e^{\lambda t} \cos \mu t + e^{\lambda t} i \sin \mu t \\ y_2 &= e^{\lambda t} \cos \mu t - e^{\lambda t} i \sin \mu t \\ y_1 + y_2 &= (2) e^{\lambda t} \cos \mu t \\ y_1 - y_2 &= (2i) e^{\lambda t} \sin \mu t \\ y &= c_1 e^{\lambda t} \cos \mu t + c_2 e^{\lambda t} \sin \mu t \end{aligned}$$

Real initial conditions will yield real c_i

Complex Conjugate Roots

- ODE: $ay'' + by' + cy$
- characteristic equation: $ar^2 + br + c$
- $b^2 - 4ac < 0$, complex conjugate roots $\lambda \pm i\mu$
- general solution: $y = c_1 e^{\lambda t} \cos \mu t + c_2 e^{\lambda t} \sin \mu t$
- any $y(t_0)$ and $y'(t_0)$ values can be satisfied by choosing c_1 and c_2
- real solutions are sinusoidal oscillations with frequency μ and damping/expansion factor $e^{\lambda t}$
- $\lambda = 0$ yields bounded oscillation

Example

$$\begin{aligned} 16y'' - 8y' + 145y = 0, \quad y(0) = -2, \quad y'(0) = 1 \\ 16r^2 - 8r + 145 = 0 \\ r = \frac{1}{4} \pm 3i \\ y = c_1 e^{0.25t} \cos 3t + c_2 e^{0.25t} \sin 3t \\ c_1 \times 1 + c_2 \times 0 = -2 \\ 0.25c_1 + 3c_2 = 1 \\ c_1 = -2 \quad c_2 = \frac{1}{2} \\ y = -2e^{0.25t} \cos 3t + \frac{1}{2} e^{0.25t} \sin 3t \end{aligned}$$

Growing oscillation.

Example

$$y'' + 9y = 0$$
$$r^2 + 9 = 0$$
$$r = \pm 3i$$
$$y = c_1 e^{0t}\cos 3t + c_2 e^{0t}\sin 3t = c_1 \cos 3t + c_2 \sin 3t$$

Bounded oscillation for any real y_0, y_0'.

Repeated Real Root

$$ay'' + by' + cy = 0$$
$$ar^2 + br + c = 0$$
$$b^2 - 4ac = 0 \to \text{repeated real root}$$
$$r_1 = r_2 = r$$
$$y_1 = e^{rt}$$
$$y_2 = te^{rt}$$

Example

$$y'' + 4y' + 4y = 0$$
$$(r+2)^2 = 0$$
$$y_1 = e^{-2t}, \quad y_2 = te^{-2t}$$

Check y_2 in ODE

$$y_2 = te^{-2t}$$
$$y_2' = -2e^{-2t}t + e^{-2t} = e^{-2t}(1-2t)$$
$$y_2'' = -2e^{-2t}(1-2t) - 2e^{-2t} = -2e^{-2t}(2-2t) = -4e^{-2t}(1-t)$$
$$y'' + 4y' + 4y = -4e^{-2t}(1-t) + 4e^{-2t}(1-2t) + 4te^{-2t}$$
$$= 4e^{-2t}\left[-1 + t + 1 - 2t + t\right] = 0$$

Example

$$y'' + 4y' + 4y = 0, \quad y(0) = 1, \quad y'(0) = -1$$
$$(r+2)^2 = 0$$
$$y_1 = e^{-2t}, \quad y_2 = te^{-2t}$$
$$y = c_1 e^{-2t} + c_2 te^{-2t} \quad \text{and} \quad y' = -2c_1 e^{-2t} - c_2 e^{-2t}(1-2t)$$

$$c_1 = 1$$
$$-2c_1 - c_2 = -1$$

$$y = e^{-2t} - te^{-2t}$$
$$y' = -2e^{-2t} + e^{-2t}(1-2t)$$

Summary

- $ay'' + by' + cy = 0$ constant coefficients
- characteristic equation: $ar^2 + br + c$
- distinct real roots r_1 and r_2

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

- complex conjugate roots $\lambda \pm i\mu$

$$y = c_1 e^{\lambda t} \cos \mu t + c_2 e^{\lambda t} \sin \mu t$$

- repeated real root r

$$y = c_1 e^{rt} + c_2 t e^{rt}$$

- Proofs of general solution forms?