

Review Set 1

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Ordinary Differential Equations

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Integrating Factors for First Order ODEs

Given $M(x, y) + N(x, y)y'$ is not exact,

If $P = \frac{(M_y - N_x)}{N}$ is a function of x only, then

$$\frac{d\mu}{dx} = P(x)\mu \rightarrow \mu(x) = \exp\left[\int P(x)dx\right]$$

is an integrating factor.

Integrating Factors for First Order ODEs

Given $M(x, y) + N(x, y)y'$ is not exact,

If $\tilde{P} = \frac{(N_x - M_y)}{M}$ is a function of y only, then

$$\frac{d\mu}{dy} = \tilde{P}(y)\mu \rightarrow \mu(y) = \exp\left[\int \tilde{P}(y)dy\right]$$

is an integrating factor.

Integrating Factors for First Order ODEs

Given $M(x, y) + N(x, y)y'$ is not exact,

If $R = \frac{(N_x - M_y)}{xM - yN}$ is a function of $z = xy$ only, then

$$\frac{d\mu}{dz} = R(z)\mu \rightarrow \mu(z) = \exp\left[\int R(z)dz\right]$$

is an integrating factor.

Problem 1.1

Textbook, p. 132, Problem 1

$$(2y - x^3) + xy' = 0, \quad x \neq 0$$

$$M = (2y - x^3), \quad N = x$$

$$M_y = 2, \quad N_x = 1$$

Problem 1.1

Textbook, p. 132, Problem 1

$$\frac{M_y - N_x}{N} = \frac{2 - 1}{x} = \frac{1}{x}$$
$$\mu(x) = e^{\int x^{-1} dx} = C e^{\ln|x|} = x$$

integrating factor: $\mu(x) = x$

Problem 1.1

Textbook, p. 132, Problem 1

$$(2xy - x^4) + x^2y' = 0, \quad x \neq 0$$

$$(\mu M)_y = (\mu N)_x = 2x$$

exact ODE

Problem 1.1

Textbook, p. 132, Problem 1

$$\Psi = \int \mu M dx = \int (2xy - x^4) dx = x^2 y - \frac{x^5}{5} + h = Q + h$$

$$\mu N = Q_y + h$$

$$x^2 = x^2 + h' \rightarrow h = C$$

$$\Psi = x^2 y - \frac{x^5}{5} = C$$

$$y = \frac{C}{x^2} - \frac{x^3}{5}$$

Problem 1.2

Textbook, p. 132, Problem 2

$$y' = \frac{1 + \cos x}{2 - \sin y}$$

$$(-\cos x - 1) + (2 - \sin y)y' = 0$$

$$M = (-\cos x - 1), \quad N = (2 - \sin y)$$

separable

$$H_1 = - \int (\cos x + 1) dx = -\sin x - x$$

$$H_2 = \int (2 - \sin y) dy = 2y + \cos y$$

$$-\sin x - x + 2y + \cos y = C$$

Problem 1.3

Textbook, p. 132, Problem 4

$$y' = 3 - 6x + y - 2xy$$

$$y' + (2x - 1)y = 3 - 6x$$

linear ODE

$$p(x) = 2x - 1, \quad g(x) = 3 - 6x$$

Problem 1.3

Textbook, p. 132, Problem 4

$$p(x) = 2x - 1, \quad g(x) = 3 - 6x$$

$$\int p dx = x^2 - x$$

$$\mu(x) = e^{x^2-x}$$

$$y = Ce^{x-x^2} + e^{x-x^2} \int e^{x^2-x} (3 - 6x) dx$$

Problem 1.3

Textbook, p. 132, Problem 4

$$y = Ce^{x-x^2} + e^{x-x^2} \int e^{x^2-x}(3-6x)dx$$

$$z = x^2 - x, \quad dz = (2x - 1)dx$$

$$y = Ce^{x-x^2} - 3e^{x-x^2} \int e^z dz$$

$$\begin{aligned} y &= Ce^{x-x^2} - 3e^{x-x^2} e^{x^2-x} \\ &= Ce^{x-x^2} - 3 \end{aligned}$$

Problem 1.4

Textbook, p. 133, Problem 12

$$y' = -y + \frac{1}{1 + e^x}$$

$$y' + y = \frac{1}{1 + e^x}$$

linear ODE

$$p(x) = 1, \quad g(x) = \frac{1}{1 + e^x}$$

$$\mu = e^x$$

Problem 1.4

Textbook, p. 133, Problem 12

$$p(x) = 1, \quad g(x) = \frac{1}{1 + e^x}, \quad \mu = e^x$$

$$y = Ce^{-x} + e^{-x} \int \frac{e^x}{1 + e^x} dx$$

$$z = e^x + 1, \quad dz = e^x dx$$

$$y = Ce^{-x} + e^{-x} \int \frac{1}{z} dz$$

$$= Ce^{-x} + e^{-x} \ln|e^x + 1|$$

$$= Ce^{-x} + e^{-x} \ln(e^x + 1)$$

Problem 1.5

Textbook, p. 133, Problem 13

$$y' = 1 + 2x + y^2 + 2xy^2 = (1 + 2x)(1 + y^2)$$

$$\frac{1}{(1 + y^2)} y' = (1 + 2x)$$

$$-(1 + 2x) + \frac{1}{(1 + y^2)} y' = 0 \quad \text{separable ODE}$$

$$H_1 = - \int (1 + 2x) dx = -x - x^2$$

$$H_2 = \int \frac{1}{(1 + y^2)} dy = \arctan y$$

$$\Psi = -x - x^2 + \arctan y = C$$

$$y = \tan(x + x^2 + C)$$

Problem 1.6

Textbook, p. 132, Problem 10

$$(x^2y + xy - y) + (x^2y - 2x^2)y' = 0$$

$$M = (x^2y + xy - y), \quad N = (x^2y - 2x^2)$$

$$M_y = x^2 + x - 1, \quad N_x = 2xy - 4x$$

not exact

Problem 1.6

Textbook, p. 132, Problem 10

Try the usual integrating factor forms:

$$\frac{M_y - N_x}{N} = \frac{x^2 + x - 1 - 2xy - 4x}{x^2y - 2x^2}$$

not a function of x only

$$\frac{N_x - M_y}{M} = \frac{2xy + 4x - x^2 - x + 1}{x^2y + xy - y}$$

not a function of y only

$$\frac{N_x - M_y}{xM - yN} = \frac{2xy + 4x - x^2 - x + 1}{(x^3y + x^2y - xy) - (x^2y^2 - 2x^2y)}$$

not a function of $z = xy$ only

Problem 1.6

Textbook, p. 132, Problem 10

Note $y = 0$ is a solution and that if $x = 0 \rightarrow$ not an ODE

Assume that $y \neq 0$ and $x \neq 0$.

Division is OK then, so take a common factor out of M and N :

$$(x^2y + xy - y) + (x^2y - 2x^2)y' = 0$$

$$x^2y \left[\left(1 + \frac{1}{x} - \frac{1}{x^2}\right) + \left(1 - \frac{2}{y}\right)y' \right] = 0$$

Problem 1.6

Textbook, p. 132, Problem 10

$$\left(1 + \frac{1}{x} - \frac{1}{x^2}\right) + \left(1 - \frac{2}{y}\right)y' = 0$$

$$\tilde{M} = \left(1 + \frac{1}{x} - \frac{1}{x^2}\right), \quad \tilde{N} = \left(1 - \frac{2}{y}\right)$$

separable ODE

$$\Psi = \int \tilde{M} dx + \int \tilde{N} dy = C$$

$$\Psi = \left(x + \ln|x| + \frac{1}{x}\right) + (y - 2 \ln|y|) = C$$

Problem 1.7

Textbook, p. 133, Problem 15

$$(e^x + 1)y' = y - ye^x$$

$$(e^x + 1)y' = y(1 - e^x)$$

$$(e^x + 1)y' + (e^x - 1)y = 0$$

$$y' + \frac{(e^x - 1)}{(e^x + 1)}y = 0 \quad \text{linear homogeneous ODE}$$

$$\frac{1}{y}y' + \frac{(e^x - 1)}{(e^x + 1)} = 0 \quad \text{separable ODE}$$

Problem 1.7

Textbook, p. 133, Problem 15

Both need

$$\begin{aligned} \int \frac{(e^x - 1)}{(e^x + 1)} dx & \quad z = \frac{x}{2}, \quad dz = 2dx \\ \int \frac{(e^x - 1)}{(e^x + 1)} dx &= 2 \int \frac{(e^{2z} - 1)}{(e^{2z} + 1)} dz \\ &= 2 \int \tanh z \, dz = 2 \ln(\cosh z) \end{aligned}$$

Problem 1.7

Textbook, p. 133, Problem 15

Using the linear ODE general solution form:

$$\mu = e^{2 \ln(\cosh z)} = (\cosh z)^2 = \left(\cosh \frac{x}{2}\right)^2$$

$$y = C\mu^{-1} = \frac{C}{\cosh^2\left(\frac{x}{2}\right)}$$

Problem 1.7

Textbook, p. 133, Problem 15

Using separable form:

$$\int \frac{(e^x - 1)}{(e^x + 1)} dx + \int \frac{1}{y} dy = C$$

$$2 \ln(\cosh z) + \ln|y| = C$$

$$\exp[2 \ln(\cosh z) + \ln|y|] = C$$

$$(\cosh z)^2 |y| = C$$

$$y = \frac{C}{(\cosh \frac{x}{2})^2}$$

Problem 1.7

Some related definitions:

$$\sinh x = \frac{1}{2}(e^x - e^{-x})$$

$$\cosh x = \frac{1}{2}(e^x + e^{-x})$$

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x} - 1}{e^{2x} + 1} = \frac{1 - e^{-2x}}{1 + e^{2x}}$$

$$\sinh x + \cosh x = e^x$$

$$\cosh^2 x - \sinh^2 x = 1$$

Problem 1.7

Some related definitions:

$$\int \sinh x \, dx = \cosh x, \quad \int \sinh \frac{x}{a} \, dx = a \cosh \frac{x}{a}$$

$$\int x \sinh x \, dx = x \cosh x - \sinh x$$

$$\int \cosh x \, dx = \sinh x, \quad \int \cosh \frac{x}{a} \, dx = a \sinh \frac{x}{a}$$

$$\int x \cosh x \, dx = x \sinh x - \cosh x$$

$$\int \tanh x \, dx = \ln \cosh x$$

Problem 1.7

Textbook, p. 133, Problem 15

Alternate separable form:

$$\int \frac{(e^x - 1)}{(e^x + 1)} dx + \int \frac{1}{y} dy = C$$
$$\frac{(e^x - 1)}{(e^x + 1)} = \frac{(e^x - 1 - e^x + e^x)}{(e^x + 1)} = \frac{(-1 - e^x + 2e^x)}{(e^x + 1)}$$
$$\int -1 + \frac{2e^x}{(e^x + 1)} dx + \int \frac{1}{y} dy = C$$
$$-x + 2 \ln(e^x + 1) + \ln|y| = C$$
$$e^{-x} (e^x + 1)^2 |y| = C$$
$$y = \frac{Ce^x}{(e^x + 1)^2}$$

which can be shown to be equivalent.

Problem 1.8

Textbook, p. 171, Problem 8

$$16y'' + 24y' + 9y = 0$$

$$16r^2 + 24r + 9 = 0$$

$$r_1 = r_2 = -\frac{3}{4}$$

$$y = c_1 e^{-3t/4} + c_2 t e^{-3t/4}$$

Problem 1.9

Textbook, p. 171, Problem 3

$$4y'' - 4y' - 3y = 0$$

$$4r^2 - 4r - 3 = 0$$

$$r_1 = -\frac{1}{2}, \quad r_2 = \frac{3}{2}$$

$$y = c_1 e^{-t/2} + c_2 e^{3t/2}$$

Problem 1.10

Textbook, p. 171, Problem 13 with modified initial conditions

$$9y'' + 6y' + 82y = 0, \quad 9r^2 + 6r + 82 = 0$$

$$r = \lambda \pm i\mu, \quad \lambda = -\frac{1}{3}, \quad \mu = 3$$

$$y = c_1 e^{\lambda t} \cos \mu t + c_2 e^{\lambda t} \sin \mu t$$

$$y = c_1 e^{-t/3} \cos 3t + c_2 e^{-t/3} \sin 3t$$

$$y' = c_1 [\lambda e^{\lambda t} \cos \mu t - \mu e^{\lambda t} \sin \mu t] + c_2 [\lambda e^{\lambda t} \sin \mu t + \mu e^{\lambda t} \cos \mu t]$$

$$y' = c_1 \left[-\frac{1}{3} e^{-t/3} \cos 3t - 3e^{-t/3} \sin 3t \right]$$

$$+ c_2 \left[-\frac{1}{3} e^{-t/3} \sin 3t + 3e^{-t/3} \cos 3t \right]$$

Problem 1.10

Textbook, p. 171, Problem 13

$$y(0) = 0, \quad y'(0) = 2$$

$$\begin{pmatrix} 1 & 0 \\ -\frac{1}{3} & 3 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \rightarrow \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{7}{9} \end{pmatrix}$$

$$y = e^{-t/3} \cos 3t + \frac{7}{9} e^{-t/3} \sin 3t$$

Problem 1.11

Textbook page 144 Problem 20

$$2y'' - 3y' + y = 0$$

$$2r^2 - 3r + 1 = 0$$

$$r_1 = 1, \quad r_2 = \frac{1}{2}$$

$$y = c_1 e^t + c_2 e^{t/2}$$

$$y' = c_1 e^t + \frac{1}{2} c_2 e^{t/2}$$

Problem 1.11

Textbook page 144 Problem 20

Applying initial conditions yields

$$y(0) = 2, \quad y'(0) = \frac{1}{2}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 2 \\ \frac{1}{2} \end{pmatrix}$$

$$\alpha = \frac{1}{2} - 1 = -\frac{1}{2} \neq 0 \rightarrow \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = -2 \begin{pmatrix} \frac{1}{2} & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

$$y = -e^t + 3e^{t/2}, \quad y' = -e^t + \frac{3}{2}e^{t/2}$$

Problem 1.11

Textbook page 144 Problem 20

To find the maximum we solve

$$y' = -e^t + \frac{3}{2}e^{t/2} = 0$$

$$e^t = \frac{3}{2}e^{t/2}$$

$$t = \ln\left(\frac{3}{2}\right) + \frac{t}{2}$$

$$t = 2 \ln \frac{3}{2} = \ln \frac{9}{4}$$

$$y\left(\ln \frac{9}{4}\right) = \frac{9}{4} = 2.25$$

Problem 1.11

Textbook page 144 Problem 20

To find the zero crossing we solve

$$y = -e^t + 3e^{t/2} = 0$$

$$e^t = 3e^{t/2}$$

$$t = \ln[3e^{t/2}] = (\ln 3) + \frac{t}{2}$$

$$\frac{t}{2} = (\ln 3)$$

$$t = 2 \ln 3 = \ln 9 \approx 2.2$$

Problem 1.12

Textbook page 155 Problem 2

$$y_1 = \cos t$$

$$y_2 = \sin t$$

$$\begin{aligned} W(y_1, y_2)(t) &= y_1 y_2' - y_2 y_1' \\ &= \cos t \cos t - \sin t(-\sin t) \\ &= \cos^2 t + \sin^2 t = 1 \neq 0 \end{aligned}$$

Therefore $\{\cos t, \sin t\}$ is a fundamental set of solutions.

Problem 1.13

Textbook page 155 Problem 3

$$y_1 = e^{-2t}$$

$$y_2 = te^{-2t}$$

$$\begin{aligned} W(y_1, y_2)(t) &= y_1 y_2' - y_2 y_1' \\ &= (e^{-2t})(e^{-2t} - 2te^{-2t}) - (te^{-2t})(-2e^{-2t}) \\ &= e^{-4t}(1 - 2t + 2t) = e^{-4t} \neq 0 \end{aligned}$$

Therefore $\{e^{-2t}, te^{-2t}\}$ is a fundamental set of solutions.

Problem 1.14

Textbook page 155 Problem 8

$$(t - 1)y'' - 3ty' + 4y = \sin t$$

$$y(-2) = 2, \quad y'(-2) = 1$$

Putting the ODE in standard form yields

$$y'' - \frac{3t}{(t-1)}y' + \frac{4}{(t-1)}y = \frac{\sin t}{(t-1)}$$

$$p(t) = -\frac{3t}{(t-1)}, \quad q(t) = \frac{4}{(t-1)}, \quad g(t) = \frac{\sin t}{(t-1)}$$

There is a discontinuity at $t = 1$. We have $t_0 = -2$ so the largest interval of continuity containing t_0 is $-\infty < t < 1$.