

MAP 2302/MAP 3305 Final Exam

Friday, 11 December 2009 7:30 AM – 9:30 AM
In-class, 102 Love Building

Closed Book. Two sheets of notes and calculators
allowed.

Show all work on these pages.

All answers must be justified for full credit.

Question	Points Possible	Points Awarded
1.	25	
2.	25	
3.	25	
4.	25	
5.	25	
6.	25	
Total Points	150	

Name: _____

The alias given on Exam 1 will be used when posting
anonymous grade list.

Problem 1(25 points)

1.a(15 points)

Find the general solution of

$$ty' + 2y = t^2, \quad t > 0 \tag{1}$$

1.b(10 points)

Solve the initial value problem using the general solution for ODE (1) with initial condition:

$$y(1) = \frac{1}{2}$$

Solution:

$$ty' + 2y = t^2, \quad y(1) = \frac{1}{2}, \quad t > 0$$

put in standard form $y' + 2t^{-1}y = t$

$$p(t) = 2t^{-1}, \quad g(t) = t$$

$$\mu(t) = \int p(t)dt = 2 \int t^{-1}dt = 2 \ln|t|$$

$$\mu(t) = e^{2 \ln|t|} = |t|^2 = t^2$$

$$y(t) = Ct^{-2} + t^{-2} \int t^2(t)dt$$

$$= Ct^{-2} + t^{-2} \int t^3 dt$$

$$= Ct^{-2} + \frac{1}{4}t^2$$

Check if ODE is satisfied:

$$y' = \frac{d}{dt} \left(Ct^{-2} + \frac{1}{4}t^2 \right)$$

$$= -2Ct^{-3} + \frac{1}{2}t$$

$$ty' + 2y = t \left(-2Ct^{-3} + \frac{1}{2}t \right) + 2 \left(Ct^{-2} + \frac{1}{4}t^2 \right) \\ = t^2$$

as desired.

Apply initial condition to set C

$$\frac{1}{2} = y(1) = C + \frac{1}{4}$$

$$C = \frac{1}{4}$$

Problem 2(25 points)

2.a(15 points)

Find the general solution of

$$y' + y^2 \sin x = 0 \tag{2}$$

2.b(10 points)

Solve the initial value problem using the general solution for ODE (2) with initial condition:

$$y(0) = 2$$

Solution:

The ODE is separable.

$$\begin{aligned}y' + y^2 \sin x &= 0 \\y^{-2} \frac{dy}{dx} + \sin x &= 0 \\ \int M(x) dx &= \int \sin x dx = -\cos x \\ \int N(y) dy &= \int y^{-2} dy = -y^{-1} \\ -\cos x - y^{-1} &= C, \quad y \neq 0\end{aligned}$$

Also $y = 0$ is a solution everywhere in the (x, y) plane.

Applying the initial condition yields

$$\begin{aligned}-\cos x - y^{-1} &= C \\ -\cos 0 - \frac{1}{2} &= -\frac{3}{2} = C\end{aligned}$$

Problem 3(25 points)

3.a(15 points)

Find the general solution of

$$(e^x \sin y - 2y \sin x) + (e^x \cos y + 2 \cos x)y' = 0 \quad (3)$$

Solution:

$$(e^x \sin y - 2y \sin x) + (e^x \cos y + 2 \cos x)y' = 0$$

$$M_y = e^x \cos y - 2 \sin x = N_x \quad \text{exact ODE}$$

$$\begin{aligned} \Psi = Q(x, y) + h(y) &= h(y) + \int (e^x \sin y - 2y \sin x) dx \\ &= h(y) + e^x \sin y + 2y \cos x \end{aligned}$$

$$N = Q_y + h'$$

$$e^x \cos y + 2 \cos x = e^x \cos y + 2 \cos x + h'$$

$$h' = 0$$

$$\Psi(x, y) = e^x \sin y + 2y \cos x = C$$

3.b(10 points)

Solve the initial value problem using the general solution for ODE (3) with initial condition:

$$y(0) = \pi$$

Solution:

$$\Psi(x, y) = e^x \sin y + 2y \cos x = C$$

$$\Psi(0, \pi) = \sin \pi + 2\pi = C$$

$$C = 2\pi$$

Problem 4(25 points)

4.a(15 points)

Find the general solution of

$$(x + 2) \sin y + (x \cos y)y' = 0 \quad (4)$$

Solution:

$$M + Ny' = (x + 2) \sin y + (x \cos y)y' = 0$$

$$M_y = (x + 2) \cos y$$

$$N_x = \cos y \quad \text{not exact}$$

Determine integrating factor μ

$$p = -(M_y - N_x)/N = -((x + 2) \cos y - \cos y)/(x \cos y) = -(x + 1)/x \quad \text{is a function of } x \text{ only}$$

$$\mu' + (-(M_y - N_x)/N)\mu = 0$$

$$\mu' + p\mu = 0$$

$$\mu' - \frac{x+1}{x}\mu = 0$$

$$-\int p = \int \frac{x+1}{x} dx = \int 1 + \frac{1}{x} dx$$

$$\mu = e^{-\int p} = xe^x$$

Apply integrating factor

$$\mu = xe^x$$

$$\begin{aligned}\mu M + \mu N y' &= (x^2 e^x + 2x e^x) \sin y + (x^2 e^x \cos y) y' = 0 \\ (\mu M)_y &= (x^2 e^x + 2x e^x) \cos y = (\mu N)_x \quad \text{exact}\end{aligned}$$

$$\begin{aligned}\Psi &= Q + h = \int (x^2 e^x + 2x e^x) \sin y dx \\ &= h + e^x [2x - 2 + x^2 - 2x + 2] \sin y = h + x^2 e^x \sin y\end{aligned}$$

$$Q_y = x^2 e^x \cos y$$

$$\begin{aligned}\mu N &= Q_y + h' \\ x^2 e^x \cos y &= h' + x^2 e^x \cos y \\ h' &= 0\end{aligned}$$

$$\Psi = Q + h = x^2 e^x \sin y = C$$

4.b(10 points)

Solve the initial value problem using the general solution for ODE (4) with initial condition:

$$y(1) = \pi/2$$

Solution:

$$\Psi(x, y) = x^2 e^x \sin y = C$$

$$\Psi(1, \pi/2) = e \sin \pi/2 = e$$

Problem 5(25 points)

5.a(15 points)

Find the general solution of

$$y'' - y' - 2y = 2e^{-t} \quad (5)$$

Solution:

$$\begin{aligned} y'' - y' - 2y &= 2e^{-t} \\ r^2 - r - 2 &= 0 \rightarrow r_1 = 2, \quad r_2 = -1 \\ \{y_1, y_2\} &= \{e^{2t}, e^{-t}\} \end{aligned}$$

Undetermined Coefficients: $g = 2e^{-t}$ is a homogeneous solution so we must apply an extra factor of t when forming Y . We have

$$\begin{aligned} Y &= Ate^{-t} \\ Y' &= Ae^{-t} - Ate^{-t} \\ Y'' &= -2Ae^{-t} + Ate^{-t} \\ Y'' - Y' - 2Y &= 2e^{-t} \\ e^{-t}[-2A - A] + te^{-t}[A + A - 2A] &= 2e^{-t} \\ e^{-t}[-3A] &= 2e^{-t} \\ A &= -\frac{2}{3} \end{aligned}$$

Therefore

$$\begin{aligned} Y &= -\frac{2}{3}te^{-t} \\ y &= -\frac{2}{3}te^{-t} + c_1e^{2t} + c_2e^{-t} \end{aligned}$$

are a particular and the general solution respectively.

Variation of Parameters:

$$\begin{aligned}y_1 &= e^{2t} \\y_2 &= e^{-t} \\g &= 2e^{-t} \\W(y_1, y_2)(t) &= y_1 y_2' - y_2 y_1' \\&= -e^{2t} e^{-t} - 2e^{-t} e^{2t} = -3e^t \neq 0 \\A_1 &= - \int_{t_0}^t \frac{y_2 g}{W(y_1, y_2)} ds \\&= \frac{1}{3} \int_{t_0}^t (e^{-s})(e^{-s})(2e^{-s}) ds \\&= \frac{2}{3} \int_{t_0}^t (e^{-3s}) ds = -\frac{2}{9} e^{-3t} + \tilde{c}_1 \\A_2 &= \int_{t_0}^t \frac{y_1 g}{W(y_1, y_2)} ds \\&= -\frac{2}{3} \int_{t_0}^t (e^{-s})(e^{2s})(e^{-s}) ds \\&= -\frac{2}{3} \int_{t_0}^t ds = -\frac{2}{3} t + \tilde{c}_2\end{aligned}$$

Therefore

$$\begin{aligned}Y &= \left(-\frac{2}{9} e^{-3t} + \tilde{c}_1\right) e^{2t} + \left(-\frac{2}{3} t + \tilde{c}_2\right) e^{-t} \\&= -\frac{2}{9} e^{-t} + \tilde{c}_1 e^{2t} + \tilde{c}_2 e^{-t} - \frac{2}{3} t e^{-t}\end{aligned}$$

is a particular solution. Since the first three terms can be absorbed into a homogeneous solution, we can take

$$\begin{aligned}Y &= -\frac{2}{3} t e^{-t} \\y &= c_1 e^{2t} + c_2 e^{-t} - \frac{2}{3} t e^{-t}\end{aligned}$$

as a particular solution and the general solution respectively.

Laplace Transforms

$$\begin{aligned}G(s) &= 2\mathcal{L}\{e^{-t}\} = \frac{2}{(s+1)} \\Y_p(s) &= \frac{G(s)}{(s-2)(s+1)} = \frac{2}{(s-2)(s+1)^2} = 2\left\{\frac{A}{(s-2)} + \frac{B}{(s+1)} + \frac{C}{(s+1)^2}\right\}\end{aligned}$$

The partial fraction has a solution

$$A = \frac{1}{9}, \quad B = -\frac{1}{9}, \quad C = -\frac{1}{3}$$

When computing the inverse Laplace transform the first two terms give homogeneous solutions so we take the last as defining the particular solution of interest

$$y_p(t) = \mathcal{L}^{-1}\left\{-\frac{2}{3} \frac{1}{(s+1)^2}\right\} = -\frac{2}{3}te^{-t}$$

as with the other methods.

5.b(10 points)

Solve the initial value problem using the general solution for ODE (5) with initial condition:

$$y(0) = 1, \quad y'(0) = 1$$

Solution:

$$y = c_1 e^{2t} + c_2 e^{-t} - \frac{2}{3} t e^{-t}$$
$$y' = 2c_1 e^{2t} - c_2 e^{-t} + \frac{2}{3} t e^{-t} - \frac{2}{3} e^{-t}$$

$$1 = c_1 + c_2$$

$$\frac{5}{3} = 2c_1 - c_2$$

$$c_1 = 8/9$$

$$c_2 = 1/9$$

Problem 6(25 points)

Find the general solution of

$$x' = Ax + g \tag{6}$$

where

$$A = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \quad g = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

Solution:

$$x' = Ax + g$$

$$A = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \quad g = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

To find the eigenpairs:

$$\det(A - \lambda I) = \lambda^2 - 1 \longrightarrow \lambda_1 = 1 \quad \lambda_2 = -1$$

$$(A - \lambda_1 I)v^{(1)} = \begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} v_1^{(1)} \\ v_1^{(1)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \longrightarrow \begin{pmatrix} v_1^{(1)} \\ v_1^{(1)} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$(A - \lambda_2 I)v^{(2)} = \begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} v_1^{(2)} \\ v_1^{(2)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \longrightarrow \begin{pmatrix} v_1^{(2)} \\ v_1^{(2)} \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

Therefore we have:

$$M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}, \quad T^{-1} = \frac{1}{2} \begin{pmatrix} 3 & -1 \\ -1 & 1 \end{pmatrix}$$

We now solve $y' = My + f$ for a particular solution vector y :

$$f = T^{-1}g = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 3 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 5 \\ 2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 13 \\ -3 \end{pmatrix}$$

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 13/2 \\ -3/2 \end{pmatrix}$$

$$y_1 = e^t \int e^{-t} \frac{13}{2} dt = -\frac{13}{2}$$

$$y_2 = -e^{-t} \int e^t \frac{3}{2} dt = -\frac{3}{2}$$

Next we transform the solution back to the original domain to get a particular solution:

$$x^{(p)} = Ty$$

$$x^{(p)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} y_1 + \begin{pmatrix} 1 \\ 3 \end{pmatrix} y_2$$

$$x^{(p)} = -\begin{pmatrix} 1 \\ 1 \end{pmatrix} \frac{13}{2} - \begin{pmatrix} 1 \\ 3 \end{pmatrix} \frac{3}{2} = -\begin{pmatrix} 8 \\ 11 \end{pmatrix}$$

To verify this is a particular solution note that

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = -\begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} 8 \\ 11 \end{pmatrix} + \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

as desired.

The general solution to the nonhomogeneous ODE is therefore

$$x(t) = c_1 e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ 3 \end{pmatrix} - \begin{pmatrix} 8 \\ 11 \end{pmatrix}$$

The particular solution can also be generated very quickly in a manner of undetermined coefficients. Given that $g(t)$ is a constant vector, one could hypothesize that a constant particular solution exists. If $x^{(p)}$ is a constant vector then its derivative is the 0 vector and

the system of linear differential equations becomes a system of linear algebraic equations:

$$x' = Ax + g \rightarrow Ax = -g$$

$$\begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = - \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

$$- \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} 8 \\ 11 \end{pmatrix} = - \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

as desired.

