Solutions for Homework 1 MAP 2302/3305 Fall 2009

Problem 1.1

1.1.a

Textbook, p. 24, Problem 1
Solution:

$$t^2\frac{d^2y}{dt^2}+t\frac{dy}{dt}+2y=\sin t$$

Highest derivative order is 2 and

$$\alpha_2(t)\frac{d^2y}{dt^2}+\alpha_1(t)\frac{dy}{dt}+\alpha_0(t)y=g(t)$$
$$\alpha_2(t)=t^2,\quad \alpha_1(t)=t,\quad \alpha_0(t)=2,\quad g(t)=\sin t$$
$$\therefore \text{linear and order 2}$$

1.1.b

Textbook, p. 24, Problem 2
Solution:

$$(1+y^2)\frac{d^2y}{dt^2}+t\frac{dy}{dt}+y=e^t$$

Highest derivative order is 2 and the term

$$(1+y^2)\frac{d^2y}{dt^2}$$

multiplies two of the y-related arguments, i.e., y^2 and y''. Therefore, nonlinear and order 2.

1.1.c

Textbook, p. 24, Problem 3
Solution:

$$\frac{d^4y}{dt^4}+\frac{d^3y}{dt^3}+\frac{d^2y}{dt^2}+\frac{dy}{dt}+y=1$$

Highest derivative order is 4 and

$$\frac{d^4y}{dt^4}+\frac{d^3y}{dt^3}+\frac{d^2y}{dt^2}+\frac{dy}{dt}+y=1$$
$$\alpha_4(t)=\alpha_3(t)=\alpha_2(t)=\alpha_1(t)=\alpha_0(t)=1,\quad g(t)=1$$
$$\therefore \text{linear and order 4}$$

1.1.d

Textbook, p. 24, Problem 4
Solution:

$$\frac{dy}{dt} + ty^2 = 0$$

Highest derivative order is 1 and the term

$$y^2$$

is a nonlinear function in y. Therefore, nonlinear and order 1.

1.1.e

Textbook, p. 24, Problem 5
Solution:

$$\frac{d^2y}{dt^2} + \sin(t+y) = \sin t$$

Highest derivative order is 2 and the term

$$\sin(t+y)$$

is a nonlinear function in y. Therefore, nonlinear and order 2.

1.1.f

Textbook, p. 24, Problem 6
Solution:

$$\frac{d^3y}{dt^3} + t\frac{dy}{dt} + (\cos^2 t)y = t^3$$

Highest derivative order is 3 and

$$\frac{d^3y}{dt^3} + t\frac{dy}{dt} + (\cos^2 t)y = t^3$$
$$\alpha_3(t) = 1, \quad \alpha_2(t) = 0, \quad \alpha_1(t) = t, \quad \alpha_0(t) = \cos^2 t, \quad g(t) = t^3$$
$$\therefore \text{ linear and order 3}$$

Problem 1.2

1.2.a

Textbook, p. 25, Problem 7
Solution:

$$y'' - y = 0$$

We have

$$\begin{aligned} y_1(t) &= e^t \\ y_1'(t) &= e^t \\ y_1''(t) &= e^t \\ \therefore y_1'' - y_1 &= 0 \end{aligned}$$

We have

$$\begin{aligned} y_2(t) &= \cosh t \\ \sinh t &= \frac{1}{2}(e^t - e^{-t}) \\ \cosh t &= \frac{1}{2}(e^t + e^{-t}) \\ (\sinh t)' &= \frac{1}{2}(e^t + e^{-t}) = \cosh t \\ (\cosh t)' &= \frac{1}{2}(e^t - e^{-t}) = \sinh t \\ (\sinh t)'' &= \sinh t \\ (\cosh t)'' &= \cosh t \\ \therefore y_2'' - y_2 &= 0 \end{aligned}$$

1.2.b

Textbook, p. 25, Problem 12
Solution:

$$t^2y'' + 5ty' + 4y = 0, \quad t > 0$$

We have

$$\begin{aligned} y_1 &= t^{-2} \\ y_1' &= -2t^{-3} \\ y_1'' &= 6t^{-4} \\ t^2y_1'' + 5ty_1' + 4y_1 &= 6t^{-2} - 10t^{-2} + 4t^{-2} = 0 \\ \therefore t^2y_1'' + 5ty_1' + 4y_1 &= 0, \quad t > 0 \end{aligned}$$

which is well-defined since $t \neq 0$.

We have

$$\begin{aligned} y_2 &= t^{-2} \ln t \\ y_2' &= t^{-3} - 2t^{-3} \ln t \\ y_2'' &= -3t^{-4} + 6t^{-4} \ln t - 2t^{-4} \\ t^2 y_2'' + 5t y_2' + 4y_2 &= t^2(-3t^{-4} + 6t^{-4} \ln t - 2t^{-4}) + 5t(t^{-3} - 2t^{-3} \ln t) + 4(t^{-2} \ln t) \\ &= -3t^{-2} + 6t^{-2} \ln t - 2t^{-2} - 10t^{-2} \ln t + 5t^{-2} + 4t^{-2} \ln t = 0 \\ \therefore t^2 y_2'' + 5t y_2' + 4y_2 &= 0, \quad t > 0 \end{aligned}$$

which is well-defined since $t > 0$.

1.2.c

Textbook, p. 25, Problem 13

Solution:

$$y'' + y = \sec t, \quad 0 < t < \pi/2$$

We have

$$\begin{aligned} y &= t(\sin t) + (\cos t) \ln(\cos t) \\ y' &= (\sin t) + t(\cos t) - (\sin t) \ln(\cos t) - \frac{(\cos t)}{(\cos t)}(\sin t) \\ &= t(\cos t) - (\sin t) \ln(\cos t) \\ y'' &= (\cos t) - t(\sin t) - (\cos t) \ln(\cos t) + \frac{(\sin t)}{(\cos t)}(\sin t) \\ y'' + y &= (\cos t) + \frac{(\sin t)}{(\cos t)}(\sin t) = \frac{1}{(\cos t)}(\cos^2 t + \sin^2 t) = \sec t \\ \therefore y'' + y &= \sec t \end{aligned}$$

The condition $0 < t < \pi/2$ guarantees the existence of the ln and sec.

1.2.d

Textbook, p. 25, Problem 14

Solution:

$$y' - 2ty = 1$$

We have

$$
\begin{aligned}
y = e^{t^2} + e^{t^2}\int_0^t e^{-s^2}ds = e^{t^2} + e^{t^2}G(t) \\
G(t)' = e^{-t^2} \\
y' = 2te^{t^2} + e^{t^2}G(t)' + 2te^{t^2}G(t) \\
= 2te^{t^2}(G(t)+1)+1 \\
y' - 2ty = 2te^{t^2}(G(t)+1)+1-2te^{t^2}(G(t)+1) = 1 \\
\therefore y' - 2ty = 1
\end{aligned}
$$

Problem 1.3

This problem uses the idea behind the method of undertermined coeffiecients to identify other simple forms of solutions to some linear ODEs. In particular, there is a polynomial whose roots define certain exponential functions as solutions to the ODE. Note, however, that the problems do not attempt to find the general solution to these ODEs, i.e., solutions parameterized by a particular number of free constants.

1.3.a

Textbook, p. 25, Problem 17
Solution:
The hypothesis to be checked is that

$$y(t) = e^{rt}$$

solves

$$y'' + y' - 6y = 0$$

for certain values of $r \in \mathbb{R}$
Substituting yields the conditions on r,

$$
\begin{aligned}
y'' + y' - 6y = (e^{rt})'' + (e^{rt})' - 6(e^{rt}) \\
= r^2(e^{rt}) + r(e^{rt}) - 6(e^{rt}) \\
= e^{rt}\big(r^2 + r - 6\big) = 0 \\
\therefore r^2 + r - 6 = 0
\end{aligned}
$$

So if e^{rt} solves the ODE then $r^2 + r - 6 = 0$. The roots of the quadratic are $r = -3$ and $r = 2$ and it is easily verified that e^{2t} and e^{-3t} are solutions to the ODE.

1.3.b

Textbook, p. 25, Problem 18
Solution:
The hypothesis to be checked is that

$$y(t) = e^{rt}$$

solves

$$y''' - 3y'' + 2y' = 0$$

for certain values of $r \in \mathbb{R}$
Substituting yields the conditions on r,

$$\begin{aligned} y''' - 3y'' + 2y' &= r^3(e^{rt}) - 3r^2(e^{rt}) + 2r(e^{rt}) \\ &= e^{rt}\big(r^3 - 3r^2 + 2r\big) = 0 \\ \therefore r^3 - 3r^2 + 2r &= r(r^2 - 3r + 2) = 0 \end{aligned}$$

The roots of the cubic are $r = 0$, $r = 1$ and $r = 2$, and it is easily verified that e^{2t}, e^t, 1 are solutions to the ODE.

1.3.c

Textbook, p. 25, Problem 19
Solution:
The hypothesis to be checked is that

$$y(t) = t^r$$

solves

$$t^2y'' + 4ty' + 2y = 0$$

for certain values of $r \in \mathbb{R}$
Substituting yields the conditions on r,

$$\begin{aligned} t^2y'' + 4ty' + 2y &= t^2(t^r)'' + 4t(t^r)' + 2t^r \\ &= r(r-1)t^2(t^{r-2}) + 4rt(t^{r-1}) + 2t^r \\ &= r(r-1)t^r + 4rt^r + 2t^r \\ (r(r-1) &+ 4r + 2)t^r = 0 \\ \therefore r(r-1) + 4r + 2 &= r^2 + 3r + 2 = 0 \end{aligned}$$

The roots of the quadratic are $r = -1$ and $r = 2$, and it is easily verified that t^{-1} and t^{-2} are solutions to the ODE.

1.3.d

Textbook, p. 25, Problem 20
Solution: The hypothesis to be checked is that

$$y(t) = t^r$$

solves

$$t^2y'' - 4ty' + 4y = 0$$

for certain values of $r \in \mathbb{R}$

Substituting yields the conditions on r,

$$\begin{aligned} t^2y'' - 4ty' + 4y &= t^2(t^r)'' - 4t(t^r)' + 4t^r \\ &= r(r-1)t^2(t^{r-2}) - 4rt(t^{r-1}) + 4t^r \\ &= r(r-1)t^r - 4rt^r + 4t^r \\ (r(r-1) - 4r + 4)t^r &= 0 \\ \therefore r(r-1) - 4r + 4 &= r^2 - 5r + 4 = 0 \end{aligned}$$

The roots of the quadratic are $r = 1$ and $r = 4$, and it is easily verified that t and t^4 are solutions to the ODE.

Problem 1.4

All of these problems are solved using an integrating factor.

1.4.a

Textbook, p. 39, Problem 13
Solution:

$$\begin{aligned} y' - y &= 2te^{2t} \\ p(t) = -1, &\quad g(t) = 2te^{2t} \\ \mu(t) = Ce^{-t} &\to \mu(t) = e^{-t} \\ y(t) &= Ce^t + e^t \int e^{-t} 2te^{2t} dt \\ &= Ce^t + e^t \int 2e^{-t}te^{2t} dt \\ \text{use integration by parts} &\quad \int rs' = rs - \int r's \\ \text{with} &\quad r = t \quad s' = e^t \\ y(t) &= Ce^t + 2e^{2t}(t-1) \end{aligned}$$

Check if ODE is satisfied:

$$\begin{aligned} y' &= \frac{d}{dt}(Ce^t + 2te^{2t} - 2e^{2t}) = Ce^t - 2e^{2t} + 4te^{2t} \\ y' - y &= (Ce^t - 2e^{2t} + 4te^{2t}) - (Ce^t + 2te^{2t} - 2e^{2t}) \\ &= 2e^{2t} \end{aligned}$$

as desired.

Apply initial condition to set C.

$$\begin{aligned} 1 = y(0) &= Ce^t + 2e^{2t}(t-1) = C - 2 \\ y(t) &= 3e^t + 2e^{2t}(t-1) \end{aligned}$$

1.4.b

Textbook, p. 39, Problem 15

Solution:

$$\begin{aligned} ty' + 2y = t^2 - t + 1, \quad y(1) &= \frac{1}{2}, \quad t > 0 \\ \text{put in standard form} \quad y' + 2t^{-1}y &= t - 1 + t^{-1} \\ p(t) = 2t^{-1}, \quad g(t) &= t - 1 + t^{-1} \\ \mu(t) = \int p(t)dt &= 2\int t^{-1}dt = 2\ln|t| \\ \mu(t) = e^{2\ln|t|} &= |t|^2 = t^2 \\ y(t) &= Ct^{-2} + t^{-2}\int t^2(t - 1 + t^{-1})dt \\ &= Ct^{-2} + t^{-2}\int (t^3 - t^2 + t)dt \\ &= Ct^{-2} + \frac{1}{4}t^2 - \frac{1}{3}t + \frac{1}{2} \end{aligned}$$

Check if ODE is satisfied:

$$\begin{aligned} y' &= \frac{d}{dt}(Ct^{-2} + \frac{1}{4}t^2 - \frac{1}{3}t + \frac{1}{2}) \\ &= -2Ct^{-3} + \frac{1}{2}t - \frac{1}{3} \\ ty' + 2y &= t(-2Ct^{-3} + \frac{1}{2}t - \frac{1}{3}) + 2(Ct^{-2} + \frac{1}{4}t^2 - \frac{1}{3}t + \frac{1}{2}) \\ &= t^2 - t + 1 \end{aligned}$$

as desired.

Apply initial condition to set C

$$\frac{1}{2} = y(1) = C + \frac{1}{4} - \frac{1}{3} + \frac{1}{2}$$
$$C = \frac{1}{12}$$

1.4.c

Textbook, p. 39, Problem 16
Solution:

$$y' + 2t^{-1}y = \frac{(\cos t)}{t^2}$$
$$g(t) = \frac{(\cos t)}{t^2}$$
$$\text{as before} \quad \mu(t) = t^2$$
$$y(t) = Ct^{-2} + t^{-2}\int t^2 \frac{(\cos t)}{t^2} dt$$
$$= t^{-2}(C + \sin t)$$
$$0 = y(\pi) = \pi^{-2}(C + 0) \to C = 0$$
$$y(t) = \frac{(\sin t)}{t^2}$$

Verify that the ODE is satisfied.

$$y' = \frac{d}{dt}(\frac{(\sin t)}{t^2}) = -2t^{-3}\sin t + t^{-2}\cos t$$
$$y' + 2t^{-1}y = (-2t^{-3}\sin t + t^{-2}\cos t) + 2t^{-1}(\frac{(\sin t)}{t^2})$$
$$= \frac{(\cos t)}{t^2}$$

as desired.

Problem 1.5

In these problems you need not draw the direction field of part (a). Do parts (b) and (c) for each problem.

1.5.a

Textbook, p. 40, Problem 24

Solution:

$$
\begin{gathered}
ty' + (t+1)y = 2te^{-t}, \quad y(1) = a, \quad t > 0 \\
y' + t^{-1}(t+1)y = 2e^{-t} \\
p(t) = t^{-1}(t+1), \quad g(t) = 2e^{-t} \\
\int p(t)dt = t + \ln|t| \\
t > 0 \to t + \ln|t| = t + \ln t, \quad \mu(t) = te^{t} \\
y(t) = Ct^{-1}e^{-t} + t^{-1}e^{-t}\int (te^{t})(2e^{-t})dt \\
= t^{-1}e^{-t}\big(t^2 + C\big)
\end{gathered}
$$

Applying $y(1) = a$ to get C as a function of a yields

$$
\begin{gathered}
a = y(1) = e^{-1}(C+1) \to C = ae - 1 \\
y(t) = t^{-1}e^{-t}\big(t^2 + C\big) \\
= te^{-t} + Ct^{-1}e^{-t} \\
= te^{-t} + (ae-1)t^{-1}e^{-t}
\end{gathered}
$$

To analyze the behavior and identify the critical value a_0 first note that independently of a the first term te^{-t} goes to 0 as $t \to 0$. So if $a_0 = e^{-1}$ then $C = 0$ and $y(t) = te^{-t}$ has this stable asymptotic behavior.

If $a_0 > e^{-1}$ then $C > 0$ and the second term $Ct^{-1}e^{-t}$ dominates as $t \to 0$. This term is positive and goes to ∞, i.e., the solution is positive and unbounded. Therefore, when $a_0 > e^{-1}$ the solution $y(t)$ is positive and unbounded as $t \to 0$.

If $a_0 < e^{-1}$ then $C < 0$ and the second term is negative and goes to $-\infty$ as $t \to 0$. Therefore, when $a_0 < e^{-1}$ the solution $y(t)$ is negative and unbounded as $t \to 0$.

1.5.b

Textbook, p. 40, Problem 25

Solution:

$$ty' + 2y = \frac{(\sin t)}{t}, \quad t < 0, \quad y(-\frac{\pi}{2}) = a$$

$$\text{put in standard form} \quad y' + 2t^{-1}y = (\sin t)t^{-2}$$

$$p(t) = 2t^{-1}, \quad g(t) = (\sin t)t^{-2}$$

$$\int p(t)dt = 2\ln|t|$$

$$\mu(t) = |t|^2 = t^2$$

$$y(t) = t^{-2}\big(C + \int t^2(\sin t)t^{-2}dt\big)$$

$$= t^{-2}\big(C - \cos t\big)$$

Applying initial conditions yields

$$a = y(-\frac{\pi}{2}) = t^{-2}\big(C - \cos t\big) \to C = \frac{a\pi^2}{4}$$

$$y(t) = t^{-2}\big(\frac{a\pi^2}{4} - \cos t\big)$$

We are interested in the behavior when $t < 0$. The analysis is split into two parts: (i) $t \to -\infty$ and (ii) $t \to 0$ from the left.

Since $C - \cos t$ is bounded for all t, it follows

$$\lim_{t \to -\infty} y(t) = 0$$

and the solution looks like a damped cosine.

So the interval $t \to 0$ from the left is the interesting one. If $a_0 = 4/\pi^2$ then $C = 1$ and applying L'Hopital twice yields

$$\lim_{t \to 0_-} y(t) = \frac{1}{2}$$

If $a_0 > 4/\pi^2$ then

$$\lim_{t \to 0_-} y(t) = \lim_{t \to 0_-} \frac{C - \cos t}{t^2} = \frac{C-1}{0} = \frac{D}{0} = \infty, \quad D > 0$$

So if $a_0 > 4/\pi^2$ then $y(t)$ becomes positively unbounded as $t \to 0$ from the left. A similar argument shows that if $a_0 < 4/\pi^2$ then $y(t)$ becomes negatively unbounded as $t \to 0$ from the left.

1.5.c

Textbook, p. 40, Problem 26

Solution:

$$(\sin t)y' + (\cos t)y = e^t, \quad y(1) = a, \quad 0 < t < \pi$$
$$y' + \frac{(\cos t)}{(\sin t)}y = \frac{e^t}{(\sin t)}$$
$$y' + (\cot t)y = \frac{e^t}{(\sin t)}$$
$$p(t) = (\cot t), \quad g(t) = \frac{e^t}{(\sin t)}$$
$$z(t) = \int p(t)dt = \int (\cot t)dt = \ln|\sin t|$$
$$\mu(t) = C(\sin t), \quad 0 < t < \pi$$

The general solution is therefore

$$\begin{aligned} y(t) &= \frac{C}{(\sin t)} + \frac{1}{(\sin t)} \int \frac{(\sin t)e^t}{(\sin t)}dt \\ &= \frac{C}{(\sin t)} + \frac{e^t}{(\sin t)} \\ &= \frac{1}{(\sin t)}\big(C + e^t\big) \end{aligned}$$

The solution as a function of $a = y(1)$ is:

$$a = y(1) = \frac{1}{(\sin 1)}\big(C + e\big)$$
$$\therefore C = a(\sin 1) - e$$
$$y(t) = \frac{1}{(\sin t)}\big(e^t - e + a(\sin 1)\big)$$

Consider the behavior of $y(t)$ as $t \to \pi$ and $t \to 0$.
We have

$$\lim_{t\to\pi} y(t) = \frac{e^\pi - e + a(\sin 1)}{0} = \frac{\tilde{C}}{0}$$

By identifying when $\tilde{C} > 0$, $\tilde{C} < 0$, and $\tilde{C} = 0$ we can determine the limiting behavior. We have

$$\tilde{a}_0 = \frac{(e - e^{\pi})}{(\sin 1)} < 0$$

$$\tilde{C} \begin{cases} < 0 & \text{if } \ a < \tilde{a}_0 \\ > 0 & \text{if } \ a > \tilde{a}_0 \\ = 0 & \text{if } \ a = \tilde{a}_0 \end{cases}$$

$$\lim_{t \to \pi} y(t) = \begin{cases} -\infty & \text{if } \ a < \tilde{a}_0 \\ \infty & \text{if } \ a > \tilde{a}_0 \\ \frac{0}{0} & \text{if } \ a = \tilde{a}_0 \end{cases}$$

Applying L'Hopital's Rule in the last case of $a = \tilde{a}_0$ we have

$$\lim_{t \to \pi} y(t) = \lim_{t \to \pi} \frac{e^t}{(\cos t)} = -e^{\pi}$$

Similar analysis for $t \longrightarrow 0$ yields

$$\lim_{t \to 0} y(t) = \frac{1 - e + a(\sin 1)}{0} = \frac{\tilde{C}}{0}$$

$$a_0 = \frac{(e - 1)}{(\sin 1)} > 0$$

$$\tilde{C} \begin{cases} < 0 & \text{if } \ a < a_0 \\ > 0 & \text{if } \ a > a_0 \\ = 0 & \text{if } \ a = a_0 \end{cases}$$

$$\lim_{t \to 0} y(t) = \frac{1 - e + a(\sin 1)}{0}$$

$$\lim_{t \to 0} y(t) = \begin{cases} -\infty & \text{if } \ a < a_0 \\ \infty & \text{if } \ a >= a_0 \\ 1 & \text{if } \ a = \frac{(e-1)}{(\sin 1)} = a_0 \end{cases}$$

Noting that $\tilde{a}_0 < 0 < a_0$ and putting these together yields:

$y(1) = a$	$y(0)$	$y(\pi)$
$a > a_0$	$+\infty$	$+\infty$
$a = a_0$	1	$+\infty$
$\tilde{a}_0 < a < a_0$	$-\infty$	$+\infty$
$a = \tilde{a}_0$	$-\infty$	$-e^{\pi}$
$a < \tilde{a}_0$	$-\infty$	$-\infty$