Homework 12 MAP 2302/3305 Fall 2009

Solutions will be posted on Friday, 4 December

Problem 12.1

12.1.a

Textbook, p. 428, Problem 1
Solution:
To find the eigenvalues:

$$A = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}$$

$$\det(A - \lambda I) = (3-\lambda)(-1-\lambda) + 4 = \lambda^2 - 2\lambda + 1$$
$$\lambda_1 = \lambda_2 = 1$$

To find the eigenvector:

$$(A - \lambda_1 I)v = 0$$

$$\begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \to \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

To find the principal vector (generalized eigenvector) for λ_1 we solve

$$(A - \lambda_1 I)p = v$$

However, since $(A - \lambda_1 I)v = 0$ there are an infinite number of such p vectors. To see this let $\mu \in \mathbb{R}$ be any real scalar and suppose $(A - \lambda_1 I)p = v$. We can create other solutions by

$$(A - \lambda_1 I)(p + \mu v) = (A - \lambda_1 I)p + \mu(A - \lambda_1 I)v = (A - \lambda_1 I)p = v$$

Therefore when we choose p we remove any piece in the direction of v. In particular we make sure that the vector p we choose has $\mu = 0$.
For this problem we have

$$(A - \lambda_1 I)p = v$$

$$\begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \to \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Note that

$$\begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

But if we happened to choose this vector we would remove any piece in the direction of v. Here we have

$$\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

i.e., $\mu = 1$. So set $\mu = 0$ and take the principal vector as

$$p = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

The general homogeneous solution is therefore

$$x(t) = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^t + c_2 \left[\begin{pmatrix} 2 \\ 1 \end{pmatrix} te^t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t \right]$$

12.1.b

Textbook, p. 428, Problem 2
Solution:
To find the eigenvalues:

$$A = \begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix}$$

$$\det(A - \lambda I) = (4 - \lambda)(-4 - \lambda) + 16 = \lambda^2$$
$$\lambda_1 = \lambda_2 = 0$$

To find the eigenvector:

$$(A - \lambda_1 I)v = 0$$

$$\begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \to \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

To find the principal vector (generalized eigenvector) for λ_1 we solve

$$(A - \lambda_1 I)p = v$$

$$\begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \to \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix}$$

The vector p has no piece in the direction of v and the general homogeneous solution is therefore

$$x(t) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \left[\begin{pmatrix} 1 \\ 2 \end{pmatrix} t + \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix} \right]$$

12.1.c

Textbook, p. 428, Problem 4
Solution:
To find the eigenvalues:

$$A = \begin{pmatrix} -3 & 5/2 \\ -5/2 & 2 \end{pmatrix}$$

$$\det(A - \lambda I) = (-3 - \lambda)(2 - \lambda) + 25/4 = \lambda^2 + \lambda + 1/4$$
$$\lambda_1 = \lambda_2 = -1/2$$

To find the eigenvector:

$$(A - \lambda_1 I)v = 0$$

$$\begin{pmatrix} -5/2 & 5/2 \\ -5/2 & 5/2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

To find the principal vector (generalized eigenvector) for λ_1 we solve

$$(A - \lambda_1 I)p = v$$

$$\begin{pmatrix} -5/2 & 5/2 \\ -5/2 & 5/2 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 2/5 \end{pmatrix}$$

The vector p has no piece in the direction of v and the general homogeneous solution is therefore

$$x(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t/2} + c_2 \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{-t/2} + \begin{pmatrix} 0 \\ 2/5 \end{pmatrix} e^{-t/2} \right]$$

Problem 12.2

12.2.a

Textbook, p. 439, Problem 1
Solution:

$$x' = Ax + g$$

$$A = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \quad g = \begin{pmatrix} e^t \\ t \end{pmatrix}$$

To find the eigenpairs:

$$\det(A - \lambda I) = \lambda^2 - 1 \longrightarrow \lambda_1 = 1 \quad \lambda_2 = -1$$

$$(A - \lambda_1 I)v^{(1)} = \begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} v_1^{(1)} \\ v_1^{(1)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \longrightarrow \begin{pmatrix} v_1^{(1)} \\ v_1^{(1)} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$(A - \lambda_2 I)v^{(2)} = \begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} v_1^{(2)} \\ v_1^{(2)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \longrightarrow \begin{pmatrix} v_1^{(2)} \\ v_1^{(2)} \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

Therefore we have:

$$M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}, \quad T^{-1} = \frac{1}{2} \begin{pmatrix} 3 & -1 \\ -1 & 1 \end{pmatrix}$$

We now solve $y' = My + f$ for a particular solution vector y:

$$f = T^{-1}g = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 3 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} e^t \\ t \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 3e^t - t \\ -e^t + t \end{pmatrix}$$

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 3e^t - t \\ -e^t + t \end{pmatrix}$$

$$y_1 = \frac{e^t}{2} \int e^{-t}(3e^t - t)dt = \frac{3}{2}te^t + \frac{1}{2}t + \frac{1}{2}$$

$$y_2 = \frac{e^{-t}}{2} \int e^t(t - e^t)dt = \frac{1}{4}e^t + \frac{1}{2}t - \frac{1}{2}$$

Next we transform the solution back to the original domain to get a particular solution:

$$x^{(p)} = Ty$$

$$x^{(p)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} y_1 + \begin{pmatrix} 1 \\ 3 \end{pmatrix} y_2$$

$$= \frac{3}{2}te^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2}t \begin{pmatrix} 1 \\ 3 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{4}e^t \begin{pmatrix} 1 \\ 3 \end{pmatrix} + \frac{1}{2}t \begin{pmatrix} 1 \\ 3 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$= \frac{3}{2}te^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{4}e^t \begin{pmatrix} 1 \\ 3 \end{pmatrix} + \frac{1}{2}t \begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The general solution to the nonhomogeneous ODE is therefore

$$x(t) = c_1 e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ 3 \end{pmatrix} + \frac{3}{2} t e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{4} e^t \begin{pmatrix} 1 \\ 3 \end{pmatrix} + \frac{1}{2} t \begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

12.2.b

Textbook, p. 439, Problem 2
Solution:

$$x' = Ax + g$$

$$A = \begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix} \qquad g = \begin{pmatrix} e^t \\ \sqrt{3} e^{-t} \end{pmatrix}$$

To find the eigenpairs:

$$\det(A - \lambda I) = \lambda^2 - 4 \longrightarrow \lambda_1 = 2 \quad \lambda_2 = -2$$

$$(A - \lambda_1 I) v^{(1)} = \begin{pmatrix} -1 & \sqrt{3} \\ \sqrt{3} & -3 \end{pmatrix} \begin{pmatrix} v_1^{(1)} \\ v_1^{(1)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \longrightarrow \begin{pmatrix} v_1^{(1)} \\ v_1^{(1)} \end{pmatrix} = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}$$

$$(A - \lambda_2 I) v^{(2)} = \begin{pmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} \begin{pmatrix} v_1^{(2)} \\ v_1^{(2)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \longrightarrow \begin{pmatrix} v_1^{(2)} \\ v_1^{(2)} \end{pmatrix} = \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix}$$

Therefore we have:

$$M = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}, \quad T = \begin{pmatrix} \sqrt{3} & 1 \\ 1 & -\sqrt{3} \end{pmatrix}, \quad T^{-1} = -\frac{1}{4} \begin{pmatrix} -\sqrt{3} & -1 \\ -1 & \sqrt{3} \end{pmatrix}$$

We now solve $y' = My + f$ for a particular solution vector y:

$$f = T^{-1} g = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = -\frac{1}{4} \begin{pmatrix} -\sqrt{3} & -1 \\ -1 & \sqrt{3} \end{pmatrix} \begin{pmatrix} e^t \\ \sqrt{3} e^{-t} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{4} e^t + \frac{\sqrt{3}}{4} e^{-t} \\ \frac{1}{4} e^t - \frac{3}{4} e^{-t} \end{pmatrix}$$

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} \frac{\sqrt{3}}{4} e^t + \frac{\sqrt{3}}{4} e^{-t} \\ \frac{1}{4} e^t - \frac{3}{4} e^{-t} \end{pmatrix}$$

$$y_1 = \frac{e^{2t} \sqrt{3}}{4} \int e^{-2t} (e^t + e^{-t}) dt = -\frac{\sqrt{3}}{4} e^t + \frac{1}{4\sqrt{3}} e^{-t}$$

$$y_2 = \frac{e^{-2t}}{4} \int e^{2t} (e^t - 3e^{-t}) dt = \frac{1}{12} e^t - \frac{3}{4} e^{-t}$$

Next we transform the solution back to the original domain to get a particular solution:

$$x^{(p)} = Ty$$

$$x^{(p)} = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix} y_1 + \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} y_2$$

$$= \begin{pmatrix} -\frac{2}{3} \\ -\frac{\sqrt{3}}{3} \end{pmatrix} e^t + \begin{pmatrix} -1 \\ \frac{2}{\sqrt{3}} \end{pmatrix} e^{-t}$$

The general solution to the nonhomogeneous ODE is therefore

$$x(t) = c_1 e^{2t} \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} - \begin{pmatrix} \frac{2}{3} \\ \frac{\sqrt{3}}{3} \end{pmatrix} e^t + \begin{pmatrix} -1 \\ \frac{2}{\sqrt{3}} \end{pmatrix} e^{-t}$$

12.2.c

Textbook, p. 440, Problem 3
Solution:

$$x' = Ax + g$$

$$A = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}, \quad g = \begin{pmatrix} -\cos t \\ \sin t \end{pmatrix}$$
$$\det(A - \lambda I) = \lambda^2 + 1 \longrightarrow \lambda_\pm = \pm i$$

Since this problem has a conjugate pair of roots we use it to exemplify the use of Laplace transforms to solve a system without any initial transformation to a special form.

In order to do this we first consider generating the Laplace transforms of some functions that tend to appear in solutions of oscillation problems, i.e., problems with complex eigenvalues. Specifically we will derive the following:

$$\mathcal{L}\{t\sin t\} = \frac{2s}{(s^2+1)^2}$$

$$\mathcal{L}\{t\cos t\} = \frac{2s^2}{(s^2+1)^2} - \frac{1}{(s^2+1)} = \frac{s^2-1}{(s^2+1)^2}$$

$$\mathcal{L}\{t\cos t + \sin t\} = \frac{2s^2}{(s^2+1)^2}$$

$$\mathcal{L}\{-t\sin t + \cos t\} = -\frac{2s}{(s^2+1)^2} + \frac{s}{(s^2+1)} = \frac{s^3-s}{(s^2+1)^2}$$

$$\mathcal{L}\{t\cos t - \sin t\} = -\frac{2s^2}{(s^2+1)^2} - \frac{2}{(s^2+1)} = -\frac{2}{(s^2+1)^2}$$

Recall that

$$\mathcal{L}\{f(t)\} = F(s)$$
$$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$$

$$\mathcal{L}\{-tf(t)\} = F'(s)$$

We have

$$\mathcal{L}\{t\sin t\} = -\mathcal{L}\{-t\sin t\} = -F'(s)$$
$$\text{where} \quad F(s) = \mathcal{L}\{\sin t\} = \frac{1}{s^2+1}$$

$$\therefore F'(s) = -\frac{2}{(s^2+1)^2} \longrightarrow \mathcal{L}\{t\sin t\} = \frac{2s}{(s^2+1)^2}$$

$$\mathcal{L}\{t\cos t\} = -\mathcal{L}\{-t\cos t\} = -F'(s)$$
$$\text{where} \quad F(s) = \mathcal{L}\{\cos t\} = \frac{s}{s^2+1}$$

$$\therefore F'(s) = -\frac{2s^2}{(s^2+1)^2} + \frac{1}{(s^2+1)} \longrightarrow \mathcal{L}\{t\cos t\} = \frac{2s^2}{(s^2+1)^2} - \frac{1}{(s^2+1)} = \frac{s^2-1}{(s^2+1)^2}$$

From these we can derive the others:

$$\mathcal{L}\{t\cos t + \sin t\} = \mathcal{L}\{t\cos t\} + \mathcal{L}\{\sin t\}$$
$$= \frac{2s^2}{(s^2+1)^2} - \frac{1}{(s^2+1)} + \frac{1}{(s^2+1)} = \frac{2s^2}{(s^2+1)^2}$$

Note also that this follows from differentiation:

$$\mathcal{L}\{(t\sin t)'\} = \mathcal{L}\{t\cos t + \sin t\} = s\mathcal{L}\{t\sin t\} - 0\sin 0$$
$$= \frac{2s^2}{(s^2+1)^2}$$

We also have

$$\mathcal{L}\{-t\sin t + \cos t\} = \mathcal{L}\{-t\sin t\} + \mathcal{L}\{\cos t\} = -\frac{2s}{(s^2+1)^2} + \frac{s}{(s^2+1)} = \frac{s^3-s}{(s^2+1)^2}$$

Note also that this follows from differentiation:

$$\mathcal{L}\{(t\cos t)'\} = \mathcal{L}\{-t\sin t + \cos t\} = s\mathcal{L}\{t\cos t\} - 0\cos 0$$
$$= s\frac{s^2-1}{(s^2+1)^2} = \frac{s^3-s}{(s^2+1)^2}$$

Finally we have

$$\mathcal{L}\{t\cos t - \sin t\} = \mathcal{L}\{t\cos t\} - \mathcal{L}\{\sin t\}$$
$$\frac{2s^2}{(s^2+1)^2} - \frac{1}{(s^2+1)} - \frac{1}{(s^2+1)} = -\frac{2}{(s^2+1)^2}$$

This also follows from integration:

$$\int -t\sin t \;\; dt = t\cos t - \sin t$$
$$\mathcal{L}\{t\cos t - \sin t\} = \mathcal{L}\{\int -t\sin t \;\; dt\} = \frac{1}{s}\mathcal{L}\{-t\sin t\}$$

Care must be taken however when using integration and differentiation with the constants of integration and the constants, e.g., $f(0)$ in the Laplace transform expressions. In the cases considered here all were 0. Also these formulas are easily generalized to have nontrivial frequencies and scales by exponentials which are also encountered in oscillation problems.

Returning to the solution of the system of ODEs using Laplace transforms we have

$$sI - A = \begin{pmatrix} s-2 & 5 \\ -1 & s+2 \end{pmatrix} \rightarrow \det(s - A) = s^2 + 1$$

$$(sI - A)^{-1} = \frac{1}{(s^2+1)} \begin{pmatrix} s+2 & -5 \\ 1 & s-2 \end{pmatrix}$$

$$g(t) = \begin{pmatrix} -\cos t \\ \sin t \end{pmatrix} \rightarrow G(s) = \begin{pmatrix} -\frac{s}{(s^2+1)} \\ \\ \frac{1}{(s^2+1)} \end{pmatrix}$$

The Laplace transform of a particular solution of the system is given by

$$X^{(p)}(s) = (sI - A)^{-1} G(s)$$

$$\begin{pmatrix} X_1^{(p)}(s) \\ X_2^{(p)}(s) \end{pmatrix} = \frac{1}{(s^2+1)} \begin{pmatrix} s+2 & -5 \\ 1 & s-2 \end{pmatrix} \begin{pmatrix} -\frac{s}{(s^2+1)} \\ \\ \frac{1}{(s^2+1)} \end{pmatrix}$$

We can recover $x_1^{(p)}(t)$ using the Laplace transforms derived above:

$$X_1^{(p)}(s) = -\frac{s^2}{(s^2+1)^2} - \frac{2s}{(s^2+1)^2} - \frac{5}{(s^2+1)^2}$$

$$= -\frac{1}{2}\frac{2s^2}{(s^2+1)^2} - \frac{2s}{(s^2+1)^2} + \frac{5}{2}\frac{(-2)}{(s^2+1)^2}$$

$$x_1^{(p)}(t) = \mathcal{L}^{-1}\{X_1^{(p)}(s)\} = 2t\cos t - 3\sin t - t\sin t$$

We can recover $x_2^{(p)}(t)$ using the Laplace transforms derived above:

$$X_2^{(p)}(s) = -\frac{2}{(s^2+1)^2}$$

$$x_2^{(p)}(t) = \mathcal{L}^{-1}\{X_2^{(p)}(s)\} = t\cos t - \sin t$$

To verify the fact that this is a particular solution to the system of ODEs we can differentiate these two solutions and compare them to

$$Ax + g$$

We have

$$x_1' = -t\cos t - 2t\sin t - \cos t - \sin t$$

$$x_2' = -t\sin t$$

and it is easily verified that

$$\begin{aligned} x_1' &= 2x_1(t) - 5x_2(t) - \cos t \\ x_2' &= x_1(t) - 2x_2(t) + \sin t \end{aligned}$$

Note that when comparing this to other particular solutions they may differ by a homogeneous solution.

12.2.d

Textbook, p. 440, Problem 6
Solution:

$$x' = Ax + g$$

$$A = \begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix} \quad g = \begin{pmatrix} t^{-1} \\ 2t^{-1} + 4 \end{pmatrix}$$
$$\det(A - \lambda I) = \lambda(\lambda + 5) \to \lambda_1 = 0, \quad \lambda_2 = -5$$

$$(A - \lambda_1 I)v^{(1)} = 0$$
$$\begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} v_1^{(1)} \\ v_2^{(1)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \to \begin{pmatrix} v_1^{(1)} \\ v_2^{(1)} \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$
$$(A - \lambda_2 I)v^{(2)} = 0$$
$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} v_1^{(2)} \\ v_2^{(2)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \to \begin{pmatrix} v_1^{(2)} \\ v_2^{(2)} \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

The general homogeneous solution is therefore

$$x_h(t) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{-5t} \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

We have therefore

$$A = \begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix} \qquad g = \begin{pmatrix} t^{-1} \\ 2t^{-1} + 4 \end{pmatrix}$$

$$T = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \qquad T^{-1} = \frac{1}{5} \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \qquad M = \begin{pmatrix} 0 & 0 \\ 0 & -5 \end{pmatrix}$$

We solve

$$y' = My + f$$

$$f = T^{-1} g$$

$$= \frac{1}{5} \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} t^{-1} \\ 2t^{-1} + 4 \end{pmatrix} = \begin{pmatrix} t^{-1} + 8/5 \\ 4/5 \end{pmatrix}$$

$$y_1' = 0y_1 + t^{-1} + 8/5$$

$$y_2' = -5y_2 + 4/5$$

So y_1 is simply computed by integrating $5t^{-1} + 8$ since the 0 eigenvalue removes y_1 from the differential equation. We have, since $t > 0$

$$y_1 = \ln t + \frac{8}{5} t$$

$$y_2 = \frac{4}{25}$$

Next we transform the solution back to the original domain to get a particular solution:

$$x^{(p)} = Ty$$

$$x^{(p)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} y_1 + \begin{pmatrix} -2 \\ 1 \end{pmatrix} y_2$$

$$= \begin{pmatrix} 1 \\ 2 \end{pmatrix} \ln t + \begin{pmatrix} 1 \\ 2 \end{pmatrix} \frac{8}{5} t + \frac{4}{25} \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

The general solution to the nonhomogeneous ODE is therefore

$$x(t) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{-5t} \begin{pmatrix} -2 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} \ln t + \begin{pmatrix} 1 \\ 2 \end{pmatrix} \frac{8t}{5} + \begin{pmatrix} \frac{-8}{25} \\ \frac{4}{25} \end{pmatrix}$$

12.2.e

Textbook, p. 440, Problem 9
Solution:

$$x' = Ax + g$$

$$A = \begin{pmatrix} -5/4 & 3/4 \\ 3/4 & -5/4 \end{pmatrix}, \quad g = \begin{pmatrix} 2t \\ e^t \end{pmatrix}$$

$$\det(A - \lambda I) = 0.25(\lambda + 8)(\lambda + 2) \to \lambda_1 = -\frac{1}{2}, \quad \lambda_2 = -2$$

Associated eigenvectors are easily seen to be

$$v^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad v^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Therefore the general homogeneous solution is

$$x_h(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t/2} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t}$$

We have

$$A = \begin{pmatrix} -5/4 & 3/4 \\ 3/4 & -5/4 \end{pmatrix}, \quad g = \begin{pmatrix} 2t \\ e^t \end{pmatrix}$$

$$M = \begin{pmatrix} -1/2 & 0 \\ 0 & -2 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad T^{-1} = -\frac{1}{2} \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}$$

$$f = T^{-1} g = -\frac{1}{2} \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 2t \\ e^t \end{pmatrix} = \begin{pmatrix} t + \frac{1}{2} e^t \\ t - \frac{1}{2} e^t \end{pmatrix}$$

The particular solutions in the y coordinate system are

$$y_1' = -\frac{1}{2}y_1 + t + \frac{1}{2}e^t$$

$$\begin{aligned} y_1 &= e^{-t/2}\int te^{t/2} + \frac{1}{2}e^{3t/2} \\ &= 2t - 4 + \frac{1}{3}e^t \end{aligned}$$

$$y_2' = -2y_2 + t - \frac{1}{2}e^t$$

$$\begin{aligned} y_2 &= e^{-2t}\int te^{2t} - \frac{1}{2}e^{3t} \\ &= \frac{1}{2}t - \frac{1}{4} - \frac{1}{6}e^t \end{aligned}$$

So we have the particular solution

$$x^{(p)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} y_1 + \begin{pmatrix} 1 \\ -1 \end{pmatrix} y_2 = \begin{pmatrix} 5/2 \\ 3/2 \end{pmatrix} t + \begin{pmatrix} 1/6 \\ 1/2 \end{pmatrix} e^t - \begin{pmatrix} 17/4 \\ 15/4 \end{pmatrix}$$

and the general solution to the nonhomogeneous problem is

$$x(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t/2} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t} + \begin{pmatrix} 5/2 \\ 3/2 \end{pmatrix} t + \begin{pmatrix} 1/6 \\ 1/2 \end{pmatrix} e^t - \begin{pmatrix} 17/4 \\ 15/4 \end{pmatrix}$$