Homework 9 MAP 2302/3305 Fall 2009

Solutions will be posted on Wednesday, 11 November

Problem 9.1

9.1.a

Textbook, p. 311, Problem 1
Solution:
$f(t)$ is continuous within each open interval and approaches a finite limit at $t = 0$, $t = 1$ and $t = 2$. It is continuous at $t = 2$. It is discontinuous at $t = 1$. Therefore $f(t)$ is piecewise continuous.

9.1.b

Textbook, p. 311, Problem 2
Solution:
$f(t)$ does not have a finite limit as $t \to 1$ from above. Therefore it is neither continuous nor piecewise continuous.

9.1.c

Textbook, p. 311, Problem 3
Solution:
$f(t)$ is continuous on each open interval and has a finite limit at the endpoints of the intervals. The functions to the left and right of $t = 1$ agree at $t = 1$. The functions to the left and right of $t = 2$ agree at $t = 2$. Therefore, $f(t)$ is continuous on $0 \leq t \leq 3$.

Problem 9.2

9.2.a

Textbook, p. 311, Problem 5
Solution:

5(a)

$$f(t) = t, \quad F(s) = \int_0^\infty te^{-st}\,dt = \lim_{A\to\infty} \int_0^A te^{-st}\,dt$$

Integrate by parts, $u = t$, $v' = e^{-st}$, $u' = 1$, $v = -e^{-st}/s$:

$$\begin{aligned}\int_0^A te^{-st}\,dt &= \left[te^{-st}\right]_0^A - \frac{1}{s^2}\left[e^{-st}\right]_0^A \\ &= Ae^{-As} - \frac{1}{s^2}\left[e^{-As} - 1\right]\end{aligned}$$

$$\lim_{A\to\infty} Ae^{-As} - \frac{1}{s^2}\left[e^{-As} - 1\right] = \frac{1}{s^2}, \quad s > 0$$

5(b)

$$f(t) = t^2, \quad F(s) = \int_0^\infty t^2 e^{-st}\,dt = \lim_{A\to\infty}\int_0^A t^2 e^{-st}\,dt$$

Integrate by parts, $u = t^2$, $v' = e^{-st}$, $u' = 2t$, $v = -e^{-st}/s$ and use the result from part (a):

$$\begin{aligned}\int_0^A t^2 e^{-st}\,dt &= \left[t^2 e^{-st}\right]_0^A + \frac{2}{s}\int_0^A te^{-st}\,dt \\ &= A^2 e^{-sA} - \frac{2A}{s}e^{-As} - \frac{2}{s^3}\left[e^{-As} - 1\right]\end{aligned}$$

$$\lim_{A\to\infty} A^2 e^{-sA} - \frac{2A}{s}e^{-As} - \frac{2}{s^3}\left[e^{-As} - 1\right] = \frac{2}{s^3}, \quad s > 0$$

5(c)

$$f(t) = t^n, \quad F(s) = \frac{n!}{s^{n+1}}$$

follows from inductively using the previous results.

Problem 9.3

9.3.a

Textbook, p. 311, Problem 15
Solution:

$$f(t) = te^{at}, \quad F(s) = \int_0^\infty te^{-(s-a)t}\,dt = \lim_{A\to\infty}\int_0^A te^{-(s-a)t}\,dt$$

Let $\tilde{s} = s - a$. Integrate by parts, $u = t$, $v' = e^{-tildest}$, $u' = 1$, $v = -e^{-\tilde{s}t}/\tilde{s}$:

$$\begin{aligned}\int_0^A te^{-\tilde{s}t}\,dt &= \left[te^{-\tilde{s}t}\right]_0^A - \frac{1}{\tilde{s}^2}\left[e^{-\tilde{s}t}\right]_0^A \\ &= Ae^{-A\tilde{s}} - \frac{1}{\tilde{s}^2}\left[e^{-A\tilde{s}} - 1\right]\end{aligned}$$

$$\begin{aligned}\lim_{A\to\infty} Ae^{-A\tilde{s}} - \frac{1}{\tilde{s}^2}\left[e^{-A\tilde{s}} - 1\right] &= \frac{1}{\tilde{s}^2}, \quad \tilde{s} > 0 \\ &= \frac{1}{s - a^2}, \quad s > a\end{aligned}$$

9.3.b

Textbook, p. 311, Problem 18

Solution:

The result follows from using the result in Problem 15 inductively. We have

$$\mathcal{L}\{t^n e^{at}\} = \frac{n!}{(s-a)^{n+1}}, \quad s > a$$

9.3.c

Textbook, p. 311, Problem 19

Solution:

$$f(t) = t^2 \sin at, \quad F(s) = \int_0^\infty t^2 e^{-st} \sin at\,dt$$

Let $u = t^2$ and $v' = e^{-st}\sin at$. We have $u' = 2t$ and

$$v = \int e^{-st} \sin at$$

which can be evaluated via integration by parts as well to get

$$\begin{aligned}v = \int e^{-st}\sin at &= -\frac{e^{-st}\sin at}{s} + \frac{a}{s}\frac{e^{-st}\cos at}{s} - \frac{a^2}{s^2}\int e^{-st}\sin at \\ \therefore v = \int e^{-st}\sin at &= -\frac{e^{-st}(s\sin at + a\cos at)}{s^2 + a^2}\end{aligned}$$

So returning to the original integration by parts we have

$$\int_0^A t^2 e^{-st}\sin at\,dt = \big[-\frac{t^2 e^{-st}(s\sin at + a\cos at)}{s^2+a^2}\big]_0^A + 2\int_0^A \frac{te^{-st}(s\sin at + a\cos at)}{s^2+a^2}\,dt$$

Taking the limit as $A \to \infty$ yields

$$F(s) = 0 + \frac{2s}{s^2+a^2}\int_0^\infty te^{-st}\sin at\;dt + \frac{2a}{s^2+a^2}\int_0^\infty te^{-st}\cos at)\;dt$$
$$= \frac{2s}{s^2+a^2}\mathcal{L}\{t\sin at\} + \frac{2a}{s^2+a^2}\mathcal{L}\{t\cos at\}$$

So we have therefore written $\mathcal{L}\{t^2\sin at\}$ in terms of $\mathcal{L}\{t\sin at\}$ and $\mathcal{L}\{t\cos at\}$. Repeating a similar integration by parts for these tranforms yields

$$\mathcal{L}\{t\sin at\} = \frac{2as}{(s^2+a^2)^2}$$
$$\mathcal{L}\{t\cos at\} = \frac{(s^2-a^2)}{(s^2+a^2)^2}$$
$$\therefore F(s) = \frac{2s}{s^2+a^2}\times\frac{2as}{(s^2+a^2)^2} + \frac{2a}{s^2+a^2}\times\frac{(s^2-a^2)}{(s^2+a^2)^2}$$

However we can also get

$$\mathcal{L}\{t\sin at\} = \frac{2as}{(s^2+a^2)^2}$$
$$\mathcal{L}\{t\cos at\} = \frac{(s^2-a^2)}{(s^2+a^2)^2}$$

by differentiation. We have

$$\mathcal{L}\{tf(t)\} = -(\mathcal{L}\{-tf(t)\}) = -F'(s)$$

So if $F(s) = \mathcal{L}\{\sin at\}$ then

$$-\mathcal{L}\{t\sin at\} = -F'(s) = -[\frac{a}{(s^2+a^2)}]'$$
$$= -[\frac{-a(s^2+a^2)'}{(s^2+a^2)^2}]$$
$$= \frac{2sa}{(s^2+a^2)^2}.$$

If $F(s) = \mathcal{L}\{\cos at\}$ then

$$-\mathcal{L}\{t\cos at\} = -F'(s) = -[\frac{s}{(s^2+a^2)}]'$$
$$-\frac{(s^2+a^2)-2s^2}{(s^2+a^2)^2} = \frac{s^2-a^2}{(s^2+a^2)^2}$$

which yields the same results as above.

Of course, the most direct way to compute $\mathcal{L}\{t^2 \sin at\}$ is via differentiation directly since if $F(s) = \mathcal{L}\{\sin at\}$ then

$$\mathcal{L}\{t^2 \sin at\} = F''(s).$$

We have

$$F(s) = \frac{a}{(s^2 + a^2)} \quad \text{and} \quad F'(s) = -\frac{2sa}{(s^2 + a^2)^2}$$

$$\begin{aligned} F''(s) &= -[\frac{2sa}{(s^2 + a^2)^2}]' \\ &= -[\frac{2a(s^2 + a^2)^2 - (2sa)(4s)(s^2 + a^2)}{(s^2 + a^2)^4}] \\ &= -[\frac{2a(a^2 - 4s^2)}{(s^2 + a^2)^3}] = \frac{2a(3s^2 - a^2)}{(s^2 + a^2)^3} \end{aligned}$$

which is the same as above.

Problem 9.4

9.4.a

Textbook, p. 311, Problem 21
Solution:

$$\begin{aligned} \int_0^\infty \frac{1}{1 + t^2} \; dt &= \lim_{A \to \infty} \int_0^A \frac{1}{1 + t^2} \; dt \\ &= \lim_{A \to \infty} \left[\arctan t\right]_0^A \\ &= \lim_{A \to \infty} \arctan A = \frac{\pi}{2} \end{aligned}$$

The integral converges.

9.4.b

Textbook, p. 311, Problem 23
Solution:

$$\int_1^\infty \frac{e^t}{t^2} \; dt = \lim_{A \to \infty} \int_1^A \frac{e^t}{t^2} \; dt$$

The function e^t/t^2 is growing so we conjecture that the improper integral diverges. To prove it note that for $t \geq 1$ we have

$$e^t > 1 + t + \frac{t^2}{2} > \frac{t^2}{2} \rightarrow \frac{e^t}{t^2} > \frac{1}{2}$$
$$\int_1^A \frac{e^t}{t^2}\ dt > \int_1^A \frac{1}{2}\ dt = \frac{A-1}{2}$$
$$\therefore \lim_{A\to\infty} \frac{A-1}{2} < \lim_{A\to\infty} \int_1^A \frac{e^t}{t^2}\ dt$$

Since $\lim_{A\to\infty}(A-1)/2 = \infty$ the integral diverges.

Problem 9.5

9.5.a

Textbook, p. 320, Problem 1
Solution:

$$F(s) = \frac{3}{s^2+4}$$

The relevant tranform is

$$\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$$

We have

$$F(s) = \frac{3}{s^2+4} = \frac{3}{2} \times \frac{2}{3}\frac{3}{s^2+4}$$
$$= \frac{3}{2}\frac{2}{s^2+4} = \frac{3}{2}\mathcal{L}^{-1}\{\frac{2}{s^2+4}\}$$

$$\therefore f(t) = \frac{3}{2}\sin 2t$$

9.5.b

Textbook, p. 320, Problem 2
Solution:

$$F(s) = \frac{4}{(s-1)^3}$$

The relevant tranform is

$$\mathcal{L}\{t^n e^{at}\} = \frac{n!}{(s-a)^{n+1}}$$

We have

$$F(s) = \frac{4}{(s-1)^3} = 2\frac{2!}{(s-1)^{2+1}}$$

$$2\mathcal{L}^{-1}\{\frac{2!}{(s-1)^{2+1}}\} = 2t^2e^t$$
$$f(t) = 2t^2e^t$$

9.5.c

Textbook, p. 320, Problem 7
Solution:

$$F(s) = \frac{2s+1}{s^2-2s+2}$$

The relevant transforms are

$$F(s-c) = \mathcal{L}\{e^{ct}f(t)\}$$
$$\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}$$
$$\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$$

We have

$$F(s) = \frac{2s+1}{s^2-2s+2} = \frac{2s+1}{(s-1)^2+1} = \frac{2(s-1)+3}{(s-1)^2+1}$$
$$= \frac{2(s-1)}{(s-1)^2+1} + \frac{3}{(s-1)^2+1}$$
$$= 2F_1(s-1) + 3F_2(s-1)$$

$$F_1(s) = \frac{s}{s^2+1}, \quad \text{and} \quad F_2(s) = \frac{1}{s^2+1}$$

Therefore, it follows that

$$f(t) = 2\mathcal{L}^{-1}\{F_1(s-1)\} + 3\mathcal{L}^{-1}\{F_2(s-1)\}$$
$$= 2e^{-t}\cos t + 3e^{-t}\sin t$$

Problem 9.6

9.6.a

Textbook, p. 320, Problem 11
Solution:

$$y'' - y' - 6y = 0, \quad y(0) = 1, \quad y'(0) = -1$$
$$a = 1, \;\; b = -1 \;\; c = -6$$

$$Y(s) = \frac{(s-1)y(0) + y'(0)}{(s^2 - s - 6} = \frac{(s-2)}{(s^2 - s - 6)}$$
$$= \frac{(s-2)}{(s-3)(s+2)} = \frac{c_1}{(s-3)} + \frac{c_2}{(s+2)}$$
$$\therefore c_1(s+2) + c_2(s-3) = s - 2$$

This yields the following linear system

$$c_1 + c_2 = 1$$
$$2c_1 - 3c_2 = -2$$

and therefore $c_1 = 1/5$ and $c_2 = 4/5$.

$$F(s) = \frac{1}{5}G(s-3) + \frac{4}{5}G(s+2)$$
$$G(s) = \frac{1}{s} \rightarrow g(t) = 1$$
$$\mathcal{L}^{-1}G(s-c) = e^{ct}g(t)$$
$$\therefore f(t) = \frac{1}{5}e^{3t} + \frac{4}{5}e^{-2t}$$

9.6.b

Textbook, p. 320, Problem 16
Solution:

$$y'' + 2y' + 5y = 0, \quad y(0) = 2, \quad y'(0) = -1$$
$$a = 1, \quad b = 2, \quad c = 5$$

$$\begin{aligned} Y(s) = \frac{(as+b)y(0) + ay'(0)}{as^2 + bs + c} &= \frac{2(s+2) - 1}{s^2 + 2s + 5} = \frac{2s+3}{s^2+2s+5} \\ &= \frac{2s+3}{(s+1)^2 + 4} = \frac{2(s+1)+1}{(s+1)^2+4} \\ &= 2F_1(s+1) + \frac{1}{2}F_2(s+1) \end{aligned}$$
$$F_1(s) = \frac{s}{s^2+4}, \quad F_2(s) = \frac{2}{s^2+4}$$
$$f_1(t) = \mathcal{L}^{-1}\{F_1(s+1)\} = e^{-t}\cos 2t f_2(t) = \mathcal{L}^{-1}\{F_2(s+1)\} = e^{-t}\sin 2t$$
$$\therefore y(t) = 2e^{-t}\cos 2t + 0.5e^{-t}\sin 2t$$

9.6.c

Textbook, p. 320, Problem 20
Solution:

$$y'' + \omega^2 y = \cos 2t$$
$$\omega^2 \neq 4, \quad y(0) = 1, \quad y'(0) = 0$$
$$a = 1, \quad b = 0, \quad c = \omega^2$$
$$F(s) = \frac{s}{s^2+4}$$

$$Y(s) = \frac{s}{s^2+\omega^2} + \frac{s}{(s^2+4)(s^2+\omega^2)}$$

A partial fraction expansion can be computed as follows:

$$Y(s) = \frac{s}{s^2+\omega^2} + \frac{s}{(s^2+4)(s^2+\omega^2)} = \frac{s}{s^2+\omega^2} + \frac{c_1 s + c_2}{(s^2+4)} + \frac{c_3 s + c_4}{(s^2+\omega^2)}$$

$$\begin{gathered} c_1 + c_3 = 0 \\ c_2 + c_4 = 0 \\ \omega^2 c_1 + 4c_3 = 1 \\ \omega^2 c_2 + 4c_4 = 0 \\ c_2 = c_4 = 0 \\ c_1 = -\frac{1}{(4-\omega^2)} \quad \text{and} \quad c_3 = \frac{1}{(4-\omega^2)} \end{gathered}$$

We therefore have

$$\begin{aligned} Y(s) &= \frac{s}{s^2+\omega^2} - \frac{1}{(4-\omega^2)}\frac{s}{(s^2+4)} + \frac{1}{(4-\omega^2)}\frac{s}{(s^2+\omega^2)} \\ &= \Big[1 + \frac{1}{(4-\omega^2)}\Big]\frac{s}{s^2+\omega^2} - \frac{1}{(4-\omega^2)}\frac{s}{(s^2+4)} \\ &= \Big(\frac{(\omega^2-5)}{(\omega^2-4)}\Big)\frac{s}{s^2+\omega^2} + \Big(\frac{1}{(\omega^2-4)}\Big)\frac{s}{s^2+4} \end{aligned}$$

Taking the inverse Laplace transform of each term yields

$$y(t) = \Big(\frac{(\omega^2-5)}{(\omega^2-4)}\Big)\cos\omega t + \Big(\frac{1}{(\omega^2-4)}\Big)\cos 2t$$

Problem 9.7

9.7.a

Textbook, p. 321, Problem 25
Solution:

$$\begin{gathered} y'' + y = f(t), \quad y(0) = y'(0) = 0 \\ a = 1, \quad b = 0, \quad c = 1 \end{gathered}$$

$$f(t) = \begin{cases} t & 0 \le t < 1 \\ 0 & t \ge 1 \end{cases}$$

We have

$$Y(s) = \frac{F(s)}{(s^2+1)}$$

We have

$$\begin{aligned}
f(t) &= t - u_1(t)t = t - u_1(t)g(t-1) \\
&\text{where} \quad g(t) = t + 1 \\
\therefore G(s) &= \frac{1}{s^2} + \frac{1}{s} \\
F(s) &= \frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-s}}{s} \\
Y(s) &= \frac{1}{s^2(s^2+1)} - \frac{e^{-s}}{s^2(s^2+1)} - \frac{e^{-s}}{s(s^2+1)}
\end{aligned}$$

Note we could get $F(s)$ directly from the definition of the Laplace transform and $f(t)$ since it becomes a definite integral.

By partial fractions

$$\begin{aligned}
\frac{1}{s^2(s^2+1)} &= \frac{1}{s^2} - \frac{1}{(s^2+1)} \\
\frac{1}{s(s^2+1)} &= \frac{1}{s} - \frac{s}{(s^2+1)} \\
Y(s) = \Big[\frac{1}{s^2} - \frac{1}{(s^2+1)}\Big] &- e^{-s}\Big[\frac{1}{s^2} - \frac{1}{(s^2+1)}\Big] - e^{-s}\Big[\frac{1}{s} - \frac{s}{(s^2+1)}\Big] \\
&= F_1(s) - e^{-s}F_1(s) - e^{-s}F_2(s)
\end{aligned}$$

Inverting the Laplace transform yields

$$\begin{aligned}
\mathcal{L}^{-1}\{F_1(s)\} &= t - \sin t = f_1(t) \\
\mathcal{L}^{-1}\{F_2(s)\} &= 1 - \cos t = f_2(t)
\end{aligned}$$

Therefore we have

$$\begin{aligned}
y(t) &= f_1(t) - u_1(t)f_1(t-1) - u_1(t)f_2(t-1) \\
&= t - \sin t - u_1(t)\big[t - \sin(t-1) - \cos(t-1)\big]
\end{aligned}$$

Problem 9.8

9.8.a

Textbook, p. 322, Problem 29
Solution:

$$\mathcal{L}\{te^{at}\} = \frac{1}{(s-a)^2}$$

can be computed directly as in Problem 15 page 311.

The result in Problem 28 page 322

$$\mathcal{L}\{-tf(t)\} = F'(s)$$

can also be used with $f(t) = e^{at}$ and

$$-\mathcal{L}\{-te^{at}\} = \Big[\frac{1}{(s-a)}\Big]' = -\Big[\frac{-1}{(s-a)^2}\Big] = \frac{1}{(s-a)^2}$$

9.8.b

Textbook, p. 322, Problem 31
Solution:

$$\mathcal{L}\{t^n e^{at}\} = \frac{n!}{(s-a)^{n+1}}, \quad s > a$$

can be computed directly as in Problem 18 page 311 and taking $a = 0$.
The result in Problem 28 page 322

$$\mathcal{L}\{(-t)^n f(t)\} = F^{(n)}(s)$$

can also be used with $f(t) = 1$ and We have

$$\mathcal{L}\{t^n\} = \Big[\frac{1}{s}\Big]' \quad n \text{ even}$$
$$\mathcal{L}\{t^n\} = -\Big[\frac{1}{s}\Big]' \quad n \text{ odd}$$

which yields

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

9.8.c

Textbook, p. 322, Problem 33
Solution:
By the result in Problem 28 page 322 we have

$$-\mathcal{L}\{-te^{at}\sin bt\} = F'(s)$$
$$f(t) = e^{at}\sin bt$$
$$F(s) = \frac{b}{(s-a)^2 + b^2}$$

Differentiating yields

$$\begin{aligned} F'(s) &= [\frac{b}{(s-a)^2+b^2}]' \\ &= -\frac{2b(s-a)}{((s-a)^2+b^2)^2} \\ \therefore \mathcal{L}\{te^{at}\sin bt\} &= \frac{2b(s-a)}{((s-a)^2+b^2)^2} \end{aligned}$$