Model Categories

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Abstract

We will define a model category and give model category structures on familiar categories, focusing on ones that have a homotopy theory. Using the devices granted by model categories, we will construct a generalized homotopy theory that aligns with the homotopy theories of the examples given above. Moreover, the generalized homotopy category will reduce to the relevant homotopy categories.

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1 Introduction

After a theory in mathematics has been developed, the most natural question is whether the theory can be generalized in a manner that will produce more results and possibly be applied to other fields in some reasonable sense. This will be our task, focusing on homotopy theory, a particularly profitable theory in Algebraic Topology, where one focuses on homotopies between continuous maps of spaces in an attempt to classify spaces. Of course, when generalizing this theory, the most appropriate category to work in is the category of topological spaces, **Top**. With the notion of homotopy we were able to define homotopy equivalence and find homotopy invariants such as the fundamental group and higher homotopy groups using path homotopies. As these results have been very beneficial, we would like to be able to generalize the theory. Furthermore, it is natural to want a category in which we can view homotopy equivalent spaces as isomorphic. For example, S^1 as a subspace of $\mathbb{R}^2$ and $\mathbb{R}^2 - 0$ are homotopy equivalent with the inclusion map i (Fig. 1) and a deformation retraction r (Fig. 2) as homotopy inverses, but in **Top**, these spaces are obviously not isomorphic. Then we will begin a discussion on the morphisms between model categories.

Fig. 1

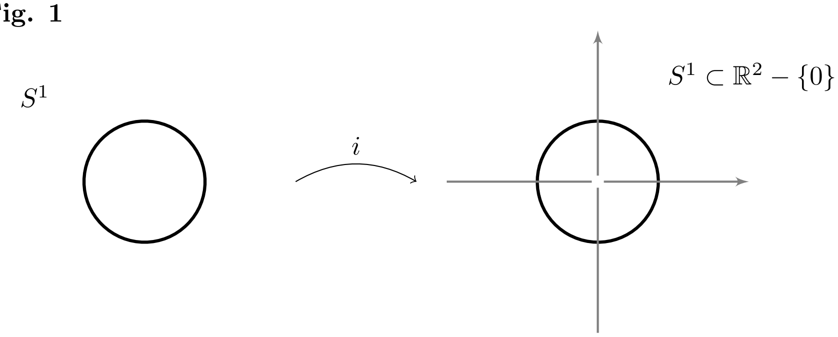
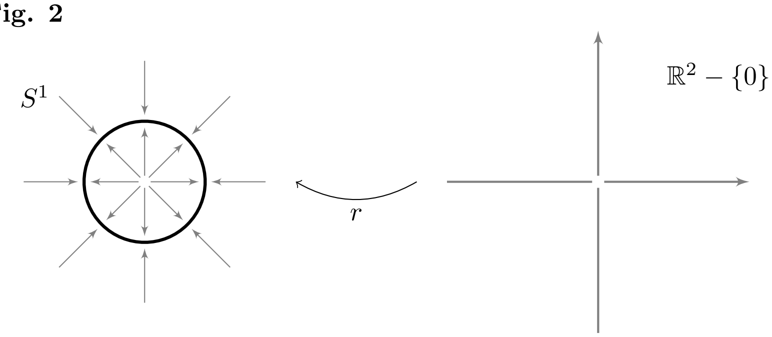


Fig. 2



Thus, we would like to have a category where these two topological spaces are isomorphic with r and i as isomorphisms i.e. $i \circ r = id_{S^1}$ and $r \circ i = id_{\mathbb{R}^2-0}$. This can be done simply by localizing **Top** at the class of morphisms of homotopy inverses which inverts all the homotopy inverses making them into isomorphisms. Thus, we have a new category where the objects remain the same, but

the morphisms are generated by the original morphisms of the category and the "extra" inverse morphisms. Unfortunately, it is not apparent that a localization of this sort is actually a locally small category at all, which for most intents and purposes is necessary. Moreover, this definition is so minimal that we do not have enough structure for calculations. This was Daniel Quillen's motivation for defining model categories which are categories with the necessary structure to give rise to a locally small category, called the homotopy category. Moreover, the homotopy category is isomorphic to the localization of our model category with respect to particular morphisms, but is well equipped for calculations. Since homotopies of continuous maps in **Top** induce homotopies of chain maps in the category of chain complexes $Ch(R)$, we have a notion of homotopy equivalence in $\mathbf{Ch}(R)$. With a suitable model structure we can form the homotopy category of $\mathbf{Ch}(R)$ which is isomorphic to the derived category $\mathrm{D}(R)$, the localization of $\mathbf{Ch}(R)$ with respect to the quasi-isomorphisms. Again, we stress that since the derived category of chain complexes is constructed by a localization, it is not obvious that $\mathrm{D}(\mathbf{Ch}(R))$ is locally small.
We will give an overview of model categories and the construction of the homotopy category paying particular attention to the examples in algebra.

2 Model Categories

The definition of a model category is quite detailed and uses a few terms from classical algebraic topology that we will review now.

2.1 Prerequisites

Let $\mathscr{C}$ be a category.

Retracts

Definition 2.1.1. [Hov99] Let $f, g \in Mor(\mathscr{C})$. Then f is a retract of g if and only if there exists a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & A \arrow[d, "f"] \\ A' \arrow[r] & B' \arrow[r] & A' \end{tikzcd}$$

such that the composition of the horizontal maps are the identity on A and A', respectively.

Liftings

Definition 2.1.2. [DS95] Let $f, g, p, i \in Mor(\mathscr{C})$. If the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & C \arrow[d, "p"] \\ B \arrow[r, "g"'] & D \end{tikzcd}$$

commutes, then a lifting with respect to f, g, p, i is a morphism $h \in Mor(\mathscr{C})$ such that the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & C \arrow[d, "p"] \\ B \arrow[r, "g"'] \arrow[ur, dashed] & D \end{tikzcd}$$

commutes.

Remark 2.1.1. [Hir03] The morphism $i : A \to B$ is said to have the *left lifting property* (LLP) with respect to p and the morphism $p : C \to D$ is said to have the *right lifting property* (RLP) with respect to i.

Functorial Factorizations

Definition 2.1.3. [Hov99] Let $\alpha, \beta \in \mathrm{Mor}(\mathscr{C})^{Mor(\mathscr{C})}$. Then (α, β) is a functorial factorization if $f = \beta(f) \circ \alpha(f)$ for all $f \in \mathrm{Mor}(\mathscr{C})$.

Base and cobase change

Definition 2.1.4. The maps $u',\ v'$ in the diagram

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "v"'] & C \arrow[d, "v'"] \\ B \arrow[r, "u'"] & D \end{tikzcd} \tag{1}$$

are called the *base change* of $u,\ v$ along $v,\ u$, respectively. Similarly, the maps $s',\ t'$ in the diagram

$$\begin{tikzcd} A \arrow[r, "s"] \arrow[d, "t"'] & C \arrow[d, "t'"] \\ B \arrow[r, "s'"] & D \end{tikzcd} \tag{2}$$

are called the *cobase change* of $s,\ t$ along $t,\ s$, respectively.

2.2 General Definition

Definition 2.2.1. [Hir03] A *model category* is a category $\mathcal{M}$ with three closed subclasses of morphisms that include identities: *weak equivalences* ($\xrightarrow{\sim}$), *fibrations* ($\twoheadrightarrow$), and *cofibrations* ($\hookrightarrow$). These subclasses must also satisfy the axioms **MC1-MC5**.

Note: An acyclic fibration (resp. cofribation) is a morphism which is a fibration (resp. cofibration) and a weak equivalence.

MC1 $\mathcal{M}$ is complete and cocomplete.

MC2 If $f, g \in$Mor($\mathcal{M}$) such that $gf \in$Mor($\mathcal{M}$) and two of the three maps are weak equivalences, then so is the third.

MC3 If f is a retract of g and g is a weak equivalence, fibration, or cofibration, then so is f, respectively.

MC4 If $f, g, i, p \in$Mor($\mathcal{M}$) such that the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & C \arrow[d, "p"] \\ B \arrow[r, "g"'] & D \end{tikzcd}$$

commutes, and i is a cofibration (resp. acyclic cofibration) and p is an acyclic fibration (resp. fibration), then there exists a lift h with respect to f, g, i, and p.

MC5 If $f \in$Mor($\mathcal{M}$), then there exists functorial factorizations (α, β) and (γ, δ) such that $\alpha(f)$ is a cofibration, $\beta(f)$ is an acyclic fibration, $\gamma(f)$ is an acyclic cofibration, and $\delta(f)$ is a fibration.

Remark 2.2.1. A model category was originally called a *"closed" model category* to emphasize it has enough structure to guarantee that any two classes of morphisms determines the third, but conveniently the "closed" has been dropped. Also, some definitions have the less stringent structure in which **MC1** only requires finite limits and colimits and the factorizations in **MC5** do not have to be functorial. In most cases, including ours, this has no effect.

Proposition 2.2.1. [DS95] Let $\mathscr{C}$ be a category, $\mathscr{D}$ be the empty category, and $F : \mathscr{D} \to \mathscr{C}$ the unique functor. Then $\overset{lim}{\longrightarrow} F$, if it exists, is an initial object of $\mathscr{C}$ and $\overset{lim}{\longleftarrow} F$, if it exists, is a terminal object of $\mathscr{C}$.

Proof. Follows directly from the definition of colimit and limit, respectively. □

Remark 2.2.2. Since $\mathcal{M}$ is complete and cocomplete, this proposition guarantees the unique existence of an initial object and terminal object in $\mathcal{M}$, denoted by $\emptyset$ and $*$, respectively.

Example 2.2.1. 1. In **Top**, the initial object is the empty set, $\emptyset$, and the terminal object is the one-point space,$*$.

2. In $\mathbf{Ch}(R)$, the initial object and the terminal object are both the zero chain complex, 0., which degree wise is the zero module. In cases when the initial object and terminal object agree, as in this case, we call the unique object the zero object.

Definition 2.2.2. [DS95] If $\emptyset \to X$ is a cofibration, then $X \in \mathcal{M}$ is a *cofibrant object*. If $X \to *$ is a fibration, then $X \in \mathcal{M}$ is a *fibrant object*.

Proposition 2.2.2. [Hov99] Let $\mathcal{M}$ be a model category.
(i) The fibrations (resp. acyclic fibrations) in $\mathcal{M}$ are the maps which have the RLP with respect to acyclic cofibrations (resp. cofibrations).
(ii) The cofibrations (resp. acyclic cofibrations) in $\mathcal{M}$ are the maps which have the LLP with respect to acyclic fibrations (resp. fibrations).

Proof. For (i), axiom **MC4** states that having the RLP is a necessary condition. Thus, we need only prove that having the RLP with respect to acyclic cofibrations (resp. cofibrations) is a sufficient condition. Suppose we have the map $f : X \to Y$ having the RLP with respect to acyclic cofibrations (resp. cofibrations). Then by axiom **MC5**, f factors as $f = p \circ i$ where $i : X \to X'$ is an acyclic cofibration (resp. cofibration) and $p : X' \to Y$ is a fibration (resp. acyclic fibration). So the diagram

$$\begin{tikzcd} X \arrow[r, "id_X"] \arrow[d, "i"'] & X \arrow[d] \\ X' \arrow[r, "p"] & Y \end{tikzcd}$$

commutes. Thus, by axiom **MC4**, there is a lift $h : X' \to X$. Since the diagram

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"'] & X' \arrow[r, "h"] \arrow[d, "p"] & X \arrow[d, "f"] \\ Y \arrow[r, "id"] & Y \arrow[r, "id"] & Y \end{tikzcd}$$

commutes, f is a retract of p. Hence, f is a fibration (resp. acyclic fibration). The argument for (ii) follows by duality. $\square$

Proposition 2.2.3. [DS95] Let $\mathcal{M}$ be a model category. Then the (acyclic) fibrations in $\mathcal{M}$ are stable under base change and the (acyclic) cofibrations are stable under cobase change.

2.3 Induced Model Categories

As one might have noticed, proving that a category is a model category is a difficult task. Thus, if we can find any shortcuts in our effort we should definitely exploit them. Some categories are constructed from others such as the dual category and pointed categories. As we will see, the model structures on these categories are induced from the category used to construct them.

Suppose $\mathcal{M}$ is a model category.

2.3.1 Dual Model Category

When discussing categories, the opposite category usually is a handy device, especially where contravariant functors appear. Thus, when using model categories one would prefer to have an easily accessible model structure for the dual category. Fortunately, this model structure follows directly.

The category $\mathcal{M}$ induces a model category structure on $\mathcal{M}^{op}$ by defining $f^{op} : Y \to X$ to be a

- *weak equivalence* if $f : X \to Y$ is a weak equivalence
- *fibration* if $f : X \to Y$ is a cofibration
- *cofibration* if $f : X \to Y$ is a fibration.

Remark 2.3.1. Amending any property that holds for $\mathcal{M}$, by simply flipping arrows and interchanging fibrations and cofibrations, will also hold for $\mathcal{M}^{op}$.

2.3.2 Comma Model Categories

These categories show up repeatedly and can be very useful. For example, the category of pointed topological spaces is a comma category constructed from **Top**. Like the dual category, if a model structure is known for the base category, a model structure follows directly for the induced pointed category.

Definition 2.3.1. Let $A \in ob(\mathscr{C})$ be fixed. Then the *coslice (or above) comma category* is the category $\mathscr{C}^A$ where the $ob(\mathscr{C}^A)$ are morphisms $A \to X$ where $X \in \mathrm{ob}\mathscr{C}$ and $hom_{\mathscr{C}^A}(A \to X, A \to Y)$ is the set of diagrams

$$\begin{tikzcd} & A \arrow[dl] \arrow[dr] & \\ X \arrow[rr] & & Y \end{tikzcd}$$

that commute.

Example 2.3.1. Letting $\mathscr{C} = $ **Top** and A be a point, the coslice comma category $\mathscr{C}^A$ is the category of pointed topological spaces.

The category $\mathcal{M}$ induces a model category structure on $\mathcal{M}^A$ by defining the commutative diagram

$$\begin{tikzcd} & A \arrow[dl] \arrow[dr] & \\ X \arrow[rr] & & Y \end{tikzcd}$$

to be a

- *weak equivalence* if $f : X \to Y$ is a weak equivalence in $\mathscr{C}$
- *fibration* if $f : X \to Y$ is a fibration in $\mathscr{C}$
- *cofibration* if $f : X \to Y$ is a cofibration in $\mathscr{C}$.

Remark 2.3.2. The slice (or under) comma category, $\mathscr{C}_A$, can be constructed in a similar manner. Moreover, $\mathscr{C}$ induces a model structure on $\mathscr{C}_A$.

2.4 Examples

Now, we give model structures on familiar categories beginning with the category **Top**. We will not prove that **Top** is a model category, but merely use it as an example to explore as we construct the homotopy category. As our main interests lie in Algebra, we will prove that $\mathbf{Ch}(R)$ is in fact a model category.

2.4.1 Model Structure I on Top

Obviously, the roman numeral I insinuates that there is a second model structure on **Top** which is in fact true and will be given below. This hints at the fact that there might exist multiple model structures for a given category, each of which will produce slightly different results. This first model structure on **Top** will represent a more classical perspective of Homotopy Theory in Algebraic Topology. Before we define the structure, we will recall a couple of devices from Algebraic Topology.

Definition 2.4.1. [DS95] A map $p \in hom_{\mathbf{Top}}(C, D)$ has the *homotopy lifting property* if for every $A \in \mathrm{ob}(\mathbf{Top})$ and every commutative diagram

$$\begin{tikzcd} A \times 0 \arrow[r] \arrow[d] & C \arrow[d, "p"] \\ A \times [0,1] \arrow[r] & D \end{tikzcd}$$

there exists a lift h.

Definition 2.4.2. [DS95] A morphism with the homotopy lifting property is a *Hurewicz fibration*.

Definition 2.4.3. [DS95] Let $A, B \in \text{ob}(\mathbf{Top})$ and $A \subset B$. Then a map $i \in hom_{\mathbf{Top}}(A, B)$ has the *homotopy extension property* if for every $Y \in \mathbf{Top}$ and commutative diagram

$$\begin{tikzcd} (B \times 0) \cup (A \times [0,1]) \arrow[r] \arrow[d] & Y \arrow[d] \\ B \times [0,1] \arrow[r] & * \end{tikzcd}$$

there exists a lift h.

Definition 2.4.4. [DS95] A map $i \in hom_{\mathbf{Top}}(A, B)$ is a *closed Hurewicz cofibration* if A is a closed subspace of B and i has the homotopy extension property.

We can define the first model category structure on **Top** [Hov99] by defining $f \in hom_{\mathbf{Top}}(X, Y)$ to be a

- *weak equivalence* if f is a homotopy equivalence
- *fibration* if f is a Hurewicz fibration
- *cofibration* if f is a closed Hurewicz cofibration.

2.4.2 Model Structure II on Top

Now, we define the more widely used model structure on **Top** where the weak equivalences are "weakened" and the fibrations are Hurcewiz fibrations, but restricted to CW-complexes. Thus, the focus is on CW-complexes.

Definition 2.4.5. [DS95] A *weak homotopy equivalence* is a map $f \in hom_{\mathbf{Top}}(X, Y)$, if for each basepoint $x \in X$ the map

$$f_* : \pi_n(X, x) \to \pi_n(Y, f(x))$$

is a bijection of pointed sets for $n = 0$ and an isomorphism of groups for $n \geq 1$.

Definition 2.4.6. A *Serre fibration* is a map $p \in hom_{\mathbf{Top}}(C, D)$, if for each CW-complex A and commutative diagram

$$\begin{tikzcd} A \times 0 \arrow[r] \arrow[d] & C \arrow[d, "p"] \\ A \times [0,1] \arrow[r] & D \end{tikzcd}$$

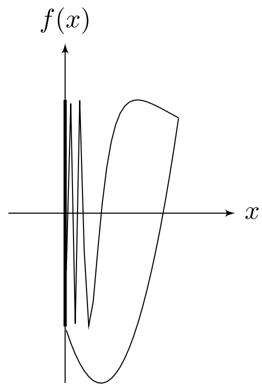
there exists a lift h.

The second model category structure on **Top** [Hov99] follows by defining a map $f \in hom_{\mathbf{Top}}(X, Y)$ to be a

- *weak equivalence* if f is a weak homotopy equivalence
- *fibration* if f is a Serre fibration
- *cofibration* if f is a retract of a map $X \to Y'$ in which Y' obtained from X by attaching cells.

Remark 2.4.1. To see that these two model structures are indeed different, notice that the morphism from the Warsaw Circle (Fig. 3) below (the subspace of $\mathbb{R}^2$ obtained by connecting the interval [-1,1] on the y-axis and the curve $sin(1/x)$ on $0 < x \leq 1$ via an arc from the point $(0, -1)$ to $(1, sin(1)))$ to a point is a weak homotopy equivalence, but not a homotopy equivalence. Thus, is a weak equivalence in (II), but not in (I).

Fig. 3



Due to the popularity of the second model structure, from this point on we will only refer to **Top** with the second model structure.

Later, it will become apparent that the fibrant objects and the cofibrant objects are the cornerstones for constructing the homotopy category. Therefore, we will go ahead and discuss these objects in **Top**.

Example 2.4.1. The fibrant objects are the objects X such that the unique map $X \to *$ is a fibration. So they are precisely the objects X where every commutative diagram of the sort

$$\begin{tikzcd} A \times 0 \arrow[r, "k"] \arrow[d] & X \arrow[d, "p"] \\ A \times [0,1] \arrow[r] & * \end{tikzcd}$$

has a lift. Since the map $k \circ \pi_0 : A \times [0,1] \to X$ where π_0 is the projection onto A is a lift for any X, every object in **Top** is fibrant.
The cofibrant objects are the objects X such that $\emptyset \to X$ is a cofibration. These are the objects for which the map $\emptyset \to X$ is a retract of a map $\emptyset \to \emptyset'$ where $\emptyset'$ is the object obtained from $\emptyset$ by attaching cells. Since the intitial object in **Top** is the empty set, $\emptyset'$ is just a cw-complex. Thus, X is a cofibrant object precisely when X is a retract of some cw-complex. Moreover, since every cw-complex is a retract of itself, all cw-complexes are cofibrant objects.

2.4.3 Chain Complexes

In order to have an idea of the generality of a model category, we step away from the topological origins for a moment and we define a model structure on a purely algebraic category, specifically, the category of nonnegatively graded chain complexes over a ring R, $\mathbf{Ch}_{\geq 0}(R)$.

We define a model category structure on $\mathbf{Ch}_{\geq 0}(R)$ by defining a map $f : X_\bullet \to Y_\bullet$ to be a

- *weak equivalence* if f is a quasi-isomorphism
- *fibration* if f is an epimorphism in positive degrees
- *cofibration* if f is a monomorphism with projective cokernel for all degrees.

Again, due to the importance of the fibrant and cofibrant objects, we will examine these objects in $\mathbf{Ch}_{\geq 0}(R)$.

Example 2.4.2. The fibrant objects are the chain complexes $C_\cdot$ such that the map $C_\cdot \to 0_\cdot$ is a fibration. Since the fibrations are just epimorphisms degree wise and every map from an R-module to the zero module is an epimorphism, every chain complex is fibrant.

As for the cofibrant objects, these are simply the chain complexes $C_\cdot$ such that the map $0_\cdot \to C_\cdot$ is a cofibration. Since the cofibrations are the monomorphisms with projective cokernels and the cokernel of the obviously injective map $0_\cdot \to C_\cdot$ is $C_\cdot$, the cofibrant objects are the chain complexes with projective R-modules in every degree.

It is worth noting that in the category of unbounded chain complexes $\mathbf{Ch}(R)$, not all chain complexes with projective R-modules in every degree are cofibrant, as the next example illustrates.

Example 2.4.3. [Hov99] Let $R = k[x]/(x^2)$, $R_\cdot$ be the chain complex with R in every degree where the differential is multiplication by x, $S^0(R)$ be the complex which is R in degree 0 and the zero module in all other degrees, and $S^0(k)$ the complex which is k in degree 0 and the zero module in all other degrees. Assume $S^0(R)$ is cofibrant. Since $R_\cdot$ is acyclic, $\emptyset \to R_\cdot$ is an acyclic cofibration. Since the natural map of $R \to k$ is a surjection, the induced map $S^0(R) \to S^0(k)$ is a fibration. The surjection $R \to k$ also induces the map $R_\cdot \to S^0(k)$ so the diagram

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d, hook, "\sim"] & S^0(R) \arrow[d, two heads] \\ R_\cdot \arrow[r] & S^0(k) \end{tikzcd}$$

commutes. With the model structure defined above on $\mathbf{Ch}(R)$, we see by MC4 there exists a lift $h : R_\cdot \to S^0(R)$, but the lift in the zeroth degree would have to be the identity on R which is not a chain map.

The following theorem will give a description of some of the chain complexes in $\mathbf{Ch}(R)$ that are cofibrant.

Theorem 2.4.1. [Hov99] Any bounded below chain complex of projective R-modules is cofibrant.

Now we will define a model structure on the category of cochain complexes $\mathbf{Ch}^{\geq 0}(R)$ by defining a map $f : X^{\cdot} \to Y^{\cdot}$ to be a

- *weak equivalence* if f induces a cohomolgy isomorphism
- *fibration* if f is an epimorphism with injective kernel for all degrees.
- *cofibration* if f is a monomorphism in positive degrees.

Before we go any further, we will actually prove that this is a model category. Although, this model category structure can be generalized to the unbounded case, we will only prove the bounded case because the unbounded case needs constructions that are beyond the scope of this paper. Moreover, the proof of the bounded case is quite long enough, as you will see.

2.5 Proof of Model Structure on $\mathbf{Ch}_{\geq 0}(R)$

2.5.1 MC1

Since R-mod is complete and cocomplete and limits and colimits of chain complexes are defined degree wise , $\mathbf{Ch}(R)$ is complete and cocomplete.

2.5.2 MC2

Suppose $C_{\cdot},\ D_{\cdot},\ E_{\cdot} \in \mathbf{Ch}_{\geq 0}(R)$ and

(i) $C_{\cdot} \xrightarrow[f]{\sim} D_{\cdot} \xrightarrow[g]{\sim} E_{\cdot}$, then the induced homomorphisms $f_*,\ g_*$ are isomorphisms. Since

$$(gf)_*(f_*^{-1}g_*^{-1}) = (g_*f_*)(f_*^{-1}g_*^{-1}) = id_C$$

and similarly

$$(f_*^{-1}g_*^{-1})(gf)_* = id_E,$$

gf is a quasi-isomorphism.

(ii) $\begin{tikzcd} C_{\cdot} \arrow[r, "\sim", "f"'] \arrow[rr, bend left, "\sim"] & D_{\cdot} \arrow[r, "g"'] & E_{\cdot} \end{tikzcd}$, then $f_*,\ (gf)_*$ are isomorphisms. As in part (i), it can be shown that $g_*^{-1} = (gf)_*^{-1} \circ f_*$. Thus, g is a quasi-isomorphism.

(iii) $\begin{tikzcd} C_{\cdot} \arrow[r, "f"'] \arrow[rr, bend left, "\sim"] & D_{\cdot} \arrow[r, "\sim", "g"'] & E_{\cdot} \end{tikzcd}$, then $g_*,\ (gf)_*$ are isomorphisms. As in part (i), it can be shown that $f_*^{-1} = g_* \circ (gf)_*^{-1}$. Thus, f is a quasi-isomorphism.

2.5.3 MC3

Suppose we have

$$\begin{tikzcd} C_. \arrow[r, "q"] \arrow[d, "f"] & C'_. \arrow[r, "r"] \arrow[d, "g"] & C_. \arrow[d, "f"] \\ D_. \arrow[r, "s"] & D'_. \arrow[r, "t"] & D_. \end{tikzcd}$$

and
(i) g is a weak equivalence. Then the diagram above induces the diagram

$$\begin{tikzcd} H(C_.) \arrow[r, "q_*"] \arrow[d, "f_*"] & H(C'_.) \arrow[r, "r_*"] \arrow[d, "g_*"] & H(C_.) \arrow[d, "f_*"] \\ H(D_.) \arrow[r, "s_*"] & H(D'_.) \arrow[r, "t_*"] & H(D_.) \end{tikzcd}$$

Since g_* is an isomorphism and the composition of the top and bottom maps are the identities on $H(C_.)$ and $H(D_.)$, respectively, it can easily be seen that $f_*^{-1} = r_* g_*^{-1} s_*$. Thus, f_* is an isomorphism. Hence, f is a quasi-isomorphism.

(ii) g is a fibration. If $h : D_. \to E_.$ such that $h_n f_n = 0$ for $n > 0$, then we have the diagram

$$\begin{tikzcd} C_n \arrow[r, "q"] \arrow[d, "f"] & C'_n \arrow[r, "r"] \arrow[d, "g"] & C_n \arrow[d, "f"] \arrow[dr] & \\ D_n \arrow[r, "s"] & D'_n \arrow[r, "t"] & D_n \arrow[d, "h"] & 0 \arrow[dl] \\ & & E_n & \end{tikzcd}$$

for $n > 0$. Since $(h_n t_n) g_n = h_n(t_n g_n) = h_n(f_n r_n) = (h_n f_n) r_n = 0$ for $n > 0$ and g_n is an epimorphism for $n > 0$, $h_n t_n = 0$. Moreover, $h_n = h_n id_{D_.} = h_n(t_n s_n) = (h_n t_n) s_n = 0 \circ r_n = 0$. Thus, f_n is an epimorphism for $n > 0$. Hence, f is a fibration.

(iii) g is a cofibration. If $h : B_. \to C_.$ such that $fh = 0$, then we have the diagram

$$\begin{tikzcd} & B_. \arrow[dl] \arrow[d, "h"] & & \\ 0 \arrow[dr] & C_. \arrow[r, "q"] \arrow[d, "f"] & C'_. \arrow[r, "r"] \arrow[d, "g"] & C_. \arrow[d, "f"] \\ & D_. \arrow[r, "s"] & D'_. \arrow[r, "t"] & D_. \end{tikzcd}$$

Since $g(qh) = (gq)h = (sf)h = s(fh) = 0$ and g is a monomorphism, $qh = 0$. Moreover, $h = id_{C_.} h = (rq)h = r(qh) = r \circ 0 = 0$. Hence, f is a monomorphism.

We also must show that it has a projective cokernel in each degree. By the universal property of cokernels we have the diagram

$$\begin{tikzcd} C_. \arrow[r, "q"] \arrow[d, "f"] & C'_. \arrow[r, "r"] \arrow[d, "g"] & C_. \arrow[d, "f"] \\ D_. \arrow[r, "s"] \arrow[d] & D'_. \arrow[r, "t"] \arrow[d] & D_. \arrow[d] \\ coker(f)_. \arrow[r, "u"] & coker(g)_. \arrow[r, "v"] & coker(f)_. \end{tikzcd}$$

Suppose $k : A_. \to B_.$ is an epimorphism and there exists a map $h : coker(f)_. \to B_.$. Since hv maps $coker(g)_.$ to $B_.$ and $coker(g)_.$ is projective, there exists a map $m : coker(g)_. \to A_.$. Thus, we have the diagram

$$\begin{tikzcd} C_. \arrow[r, "q"] \arrow[d, "f"] & C'_. \arrow[r, "r"] \arrow[d, "g"] & C_. \arrow[d, "f"] \\ D_. \arrow[r, "s"] \arrow[d] & D'_. \arrow[r, "t"] \arrow[d] & D_. \arrow[d] \\ coker(f)_. \arrow[r, "u"] & coker(g)_. \arrow[r, "v"] \arrow[d, "m"] & coker(f)_. \arrow[d, "h"] \\ & A_. \arrow[r, two heads, "k"] & B_. \end{tikzcd}$$

Since $ts = id_{D_.}$, $vu = id_{coker(f)_.}$. Thus, $hvu = h$ and $mu : coker(f)_. \to A_.$ is the desired lift. Hence, $coker(f)_.$ is projective in each degree.

2.5.4 MC4

(i) Suppose we have the diagram

$$\begin{tikzcd} A_. \arrow[r, "g"] \arrow[d, hook, "i"'] & C_. \arrow[d, two heads, "p", "\sim"'] \\ B_. \arrow[r, "h"] & D_. \end{tikzcd}$$

Since p is a fibration, p_k is an epimorphism for $k > 0$. We now show that since p is also a weak equivalence, p_0 is actually an epimorphism. Since p is a weak equivalence, p_0 is a quasi-isomorphism. Thus, we have the diagram

$$\begin{tikzcd} C_1 \arrow[r, "d_1"] \arrow[d, "p_1"] & C_0 \arrow[r] \arrow[d, "p_0"] & C_0/Im(d_1) \arrow[r] \arrow[d, "p_{0*}"] & 0 \arrow[d] \\ D_1 \arrow[r] & D_0 \arrow[r] & D_0/Im(d_1) \arrow[r] & 0 \end{tikzcd}$$

where p_1, p_{0*} are epimorphisms and the zero map is a monomorphism. Thus, the five lemma implies that p_0 is an epimorphism. Moreover,

$$0 \longrightarrow \ker p \longrightarrow C_. \longrightarrow D_. \longrightarrow 0$$

is exact. Thus, we are guaranteed a long exact sequence of homology groups. Since $H(C_.) \cong H(D_.)$, $H(\ker p) \cong 0$ i.e. the chain complex $\ker p$ is acyclic. We will use this fact shortly. To prove **MC4** (i), we must construct a chain map that lifts the diagram above. To do this, we first construct a map $f_0 : B_0 \to C_0$ and then use induction to define f_n. Since i_0 is a cofibration and $P_0 := coker(i_0)$ is projective, we have the diagram

$$\begin{tikzcd}
A_0 \arrow[r, "g_0"] \arrow[d, hook, "i_0"'] & C_0 \arrow[d, two heads, "p_0", "\sim"'] \\
A_0 \oplus P_0 \arrow[r, "h_0"] & D_0
\end{tikzcd}$$

Since P_0 naturally maps into $A_0 \oplus P_0$, composition with h_0 is a map into D_0. Since p_0 is an epimorphism and P_0 is projective, there exists a map $l_0 : P_0 \to C_0$. Thus, we have the lift

$$\begin{tikzcd}
A_0 \arrow[r, "g_0"] \arrow[d, hook, "i_0"'] & C_0 \arrow[d, two heads, "p_0", "\sim"'] \\
A_0 \oplus P_0 \arrow[r, "h_0"] \arrow[ur, dashed, "f_0"] & D_0
\end{tikzcd}$$

where $f_0 = g_0 \oplus l_0$. For $0 < k < n$, assume f_k has the properties

1. $\partial f_k = f_{k-1}\partial$
2. $p_k f_k = h_k$
3. $f_k i_k = g_k$

Now, construct $\tilde{f}_n$ the same way as f_0. Notice that $\tilde{f}_n$ has the properties 2 and 3, but not necessarily property 1. Define $\epsilon : B_n \to C_{n-1}$ by $\epsilon = \partial\tilde{f}_n - f_{n-1}\partial$. We show that ϵ induces a map $\epsilon^* : P_n \to Z_{n-1}(\ker p)$. Since $p_{n-1}\epsilon = p_{n-1}(\partial\tilde{f}_n - f_{n-1}\partial) = p_{n-1}\partial\tilde{f}_n - p_{n-1}f_{n-1}\partial = \partial p_n\tilde{f}_n - h_{n-1}\partial = \partial p_n\tilde{f}_n - \partial h_n = 0$, by the universal property of kernels there exists a map ϵ_1 such that the diagram

$$\begin{tikzcd}
 & \ker p_{n-1} \arrow[d, "j_{n-1}"] \\
B_n \arrow[ur, "\epsilon_1"] \arrow[r, "\epsilon"] & C_{n-1} \arrow[d, "p_{n-1}"] \\
 & D_{n-1}
\end{tikzcd}$$

commutes. Since $\epsilon i_n = (\partial\tilde{f}_n - f_{n-1}\partial)i_n = \partial\tilde{f}_n i_n - f_{n-1}\partial i_n = \partial g_n - f_{n-1}i_{n-1}\partial = g_{n-1}\partial - g_{n-1}\partial = 0$, $\epsilon = j_{n-1}\epsilon_1$ and j_{n-1} is injective, $\epsilon_1 i_n = 0$. Since P_n is the cokernel of i_n, by the universal property of cokernels there exists ϵ_2 such that the diagram

$$\begin{tikzcd}
A_n \arrow[d, "i_n"] & \\
B_n \arrow[r, "\epsilon_1"] \arrow[d, "\pi_n"'] & \ker p_{n-1} \\
P_n \arrow[ur, "\epsilon_2"] &
\end{tikzcd}$$

commutes. By the induction hypothesis, $\partial f_{n-1} = f_{n-2}\partial$. So $\partial\epsilon = \partial(\partial\tilde{f}_n - f_{n-1}\partial) = 0 - (\partial f_{n-1})\partial = -f_{n-2}\partial\partial = 0$. Since $\epsilon_1 = \epsilon_2\pi_n$, $j_{n-2}\partial\epsilon_2\pi_n = \partial j_{n-1}\epsilon_2\pi_n = \partial j_{n-1}\epsilon_1 = \partial\epsilon = 0$. Since j_{n-2} is injective and π_n is surjective, $\partial\epsilon_2 = 0$. Thus, $\epsilon_2 : P_n \to Z_{n-1}(\ker p)$. Since $\ker p$ is acyclic, the map $\ker p_n \to Z_{n-1}(\ker p)$ is a surjection. Moreover, there exists a map ϵ^* such that the diagram

$$\begin{tikzcd}
& \ker p_n \arrow[d, two heads] \\
P_n \arrow[ur, "\epsilon^*"] \arrow[r, "\epsilon_2"] & Z_{n-1}(\ker p)
\end{tikzcd}$$

since P_n is projective. Finally, let $f_n = \tilde{f}_n - \epsilon^*$. Since ϵ^* maps into the kernel of p_n, ϵ^* does not affect property 2. Since i_n injects onto the direct summand A_n, ϵ^* does not affect property 3. By construction, it is now clear that f_n satisfies all three properties above. Hence, we have constructed our lift.

MC4 (ii) Before we can prove this axiom we must introduce some intresting terminology and prove a couple lemmas.

Definition 2.5.1. The *n-disk chain complex* of a R-module A is defined by

$$D^n(A)_k = \begin{cases} 0 & k \neq n,\ n-1 \\ A & k = n,\ n-1 \end{cases}$$

for $n \geq 1$ where the boundary map is the identity in the nth degree and the zero map every where else.

Although, we will not need this next object at the moment, we will go ahead and define it due to its direct relation to the n-disk chain complex.

Definition 2.5.2. The *n-sphere chain complex* of a R-module A is defined by

$$S^n(A)_k = \begin{cases} 0 & k \neq n \\ A & k = n \end{cases}$$

for $n \geq 0$.

Lemma 2.5.1. Let $A \in R - mod$ and $M_\cdot \in \mathbf{Ch}(R)$. Then

$$hom_{\mathbf{Ch}(R)}(D^n(A), M) \xrightarrow{\sim} hom_R(A, M_n)$$

under the map $f \mapsto f_n$.

Proof. The map $f \mapsto f_n$ is easily seen to be bijective. □

Corollary 2.5.1. If A is projective, then

$$hom_{\mathbf{Ch}(R)}(D^n(A), M) \cong hom_R(A, M_n) \to hom_R(A, N_n) \cong hom_{Ch(R)}(D^n(A), N)$$

is surjective.

Lemma 2.5.2. Suppose $P_\cdot \in \mathbf{Ch}(R)$ is acyclic with P_n projective. Then $Z_n(P_\cdot)$ is projective and $P_\cdot \cong \oplus_{n \geq 1} D^n(Z_{n-1}(P_\cdot))$.

Proof. For $k \geq 1$ let $P^{(k)}$ be the subcomplex of $P_\cdot$ such that $P_n^{(k)}$ is P_n if $n \geq k$, $B_{n-1}(P_\cdot)$ if $n = k-1$ and 0 if $n < k-1$. Since $P_\cdot$ is acyclic, $B_n(P_\cdot) \cong Z_n(P_\cdot)$. Moreover, $P_n/B_n(P_\cdot) \cong P_n/Z_n(P_\cdot) \cong B_{n-1}(P_\cdot)$, by the first isomorphism theorem. Thus,

$$\begin{aligned}
P^{(n)}/P^{(n+1)} &\cong \ldots \to P^{(n+1)}/P^{(n+1)} \to P^{(n)}/B_n(P_\cdot) \to B_{n-1}(P_\cdot)/0 \to 0 \to \ldots \\
&\cong \ldots \to 0 \to B_{n-1}(P_\cdot) \to B_{n-1}(P_\cdot) \to 0 \to \ldots \\
&\cong \ldots \to 0 \to Z_{n-1}(P_\cdot) \to Z_{n-1}(P_\cdot) \to 0 \to \ldots \\
&\cong D^n(Z_{n-1}(P_\cdot))
\end{aligned}$$

Since $P_\cdot$ is acyclic, $P_0 = Z_0(P_\cdot)$ and

$$\begin{aligned}
0 \to Z_1(P_\cdot) \to P_1 \to P_0 \to 0 \\
= 0 \to B_1(P_\cdot) \to P_1 \to Z_0(P_\cdot) \to 0
\end{aligned}$$

is exact. Since $P_0 = Z_0(P_\cdot)$ is projective, $P_1 \cong B_1(P_\cdot) \oplus Z_0(P_\cdot)$. Thus,

$$\begin{aligned}
P &= \ldots \to P_2 \to P_1 \to P_0 \to 0 \to \ldots \\
&= \ldots \to P_2 \to P_1 \to Z_0(P_\cdot) \to 0 \to \ldots \\
&= \ldots \to P_2 \to B_1(P_\cdot) \oplus Z_0(P_\cdot) \to Z_0(P_\cdot) \to 0 \to \ldots \\
&\cong P^{(2)} \oplus D^1(Z_0(P_\cdot))
\end{aligned}$$

where $D^1(Z_0(P_\cdot))$ has a projective module in each degree. Since any direct factor of a projective R-module is projective, $P^{(2)}$ is projective in each degree. Moreover, $P^{(2)}$ is acyclic and 0 in degree zero. So we can repeat the argument above for $P^{(2)}$, but starting in degree one. Thus, $P^{(2)} \cong P^{(3)} \oplus D^2(Z_1(P_\cdot))$. Repeating in this way we will construct the desired factorization of $P_\cdot$. □

Now to prove **MC4** (ii), suppose we have the commutative diagram

$$\begin{tikzcd} A_. \arrow[r, "g"] \arrow[d, hook, "i"', "\sim"] & C_. \arrow[d, two heads, "p"] \\ B_. \arrow[r, "h"] & D_. \end{tikzcd}$$

Let $P_. = coker(i)$. Since

$$0 \to A_. \to B_. \to P_. \to 0$$

is a short exact sequence of complexes, we get a long exact sequence of homology. Since A and B are quasi-isomorphic, the long exact sequence shows that $P_.$ is acyclic. So by the previous lemma, we can write $P_. \cong \oplus_{n\geq 1} D^n(Z_{n-1}(P_.))$ where $Z_{n-1}(P_.)$ is projective and $D^n(Z_{n-1}(P_.))$ is a projective in each degree. Since $B_. \cong A_. \oplus P_.$ and p is a fibration, there exists a lift l such that the diagram

$$\begin{tikzcd} & & C_. \arrow[d, two heads, "p"] \\ P_. \arrow[r] \arrow[urr, dashed, "l"] & A_. \oplus P_. \arrow[r, "h"] & D_. \end{tikzcd}$$

commutes, by the corrollary of the first lemma. Hence, $g \oplus l$ is our desired lift.

2.5.5 MC5

(i)

Lemma 2.5.3. The map $q : Q_. \to N_.$ is a fibration if and only if q has the RLP with respect to the maps $0 \to D^n(R)$ for $n > 0$.

Proof. Recall from the proof of **MC4** (ii),

$$hom_{\mathbf{Ch}(R)}(D^n(R), N_.) \cong hom_R(R, N_n) \cong N_n.$$

The lemma follows directly. ☐

Define $P(N_.) = \oplus_{n>0} \oplus_{n\in N_n} D^n(R)$ and $p : P(N_.) \to N_.$ as the evaluation map. Then the diagram

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & P(N_.) \arrow[d, "p"] \\ D^n(R) \arrow[r] \arrow[ur, "l"] & N_. \end{tikzcd}$$

commutes where l is the natural map into the direct sum. By the lemma above, p is a fibration. Since p is an epimorphism in each degree, the map $f \oplus p$ such that the diagram

$$\begin{tikzcd}
M \arrow[dr, "i"] \arrow[drr, bend left, "f"] & & \\
& M_\cdot \oplus P(N_\cdot) \arrow[r, dashed, "f\oplus p"] & N_\cdot \\
P(N_\cdot) \arrow[ur] \arrow[urr, bend right, "p"'] & &
\end{tikzcd}$$

commutes is an epimorphism in each degree. Since

$$\begin{aligned}
H_n(M \oplus P(N_\cdot)) &\cong H_n(M) \oplus H_n(P(N_n)) \\
&\cong H_n(M) \oplus H_n(\oplus_{n>0} \oplus_{n \in N_n} D^n(R)) \\
&\cong H_n(M) \oplus [\oplus_{n>0} \oplus_{n \in N_n} H_n(D^n(R))] \\
&\cong H_n(M)
\end{aligned}$$

and the natural map i in the diagram above is by definition a monomorphism with projective cokernel, i is an acyclic cofibration. Hence, for every morphism $f : M \to N$ we have the factorization

$$\begin{tikzcd}
M_\cdot \arrow[r, hook, "\sim", "i"'] \arrow[rr, bend left, "f"] & M_\cdot \oplus P(N_\cdot) \arrow[r, two heads, "p"'] & N_\cdot
\end{tikzcd}$$

MC5 (ii) Suppose we have the map $f : M_\cdot \to N_\cdot$.

Lemma 2.5.4. [GS06] The map f is an acyclic fibration if and only if

$$M_n \to Z_{n-1}(M_\cdot) \times_{Z_{n-1}(N_\cdot)} N_n$$

is an epimorphism for $n \geq 0$.

By the universal property of fiber products, we have the commutative diagram

$$\begin{tikzcd}
& & N_n \arrow[dr] & \\
M_n \arrow[r, dashed, "h"] \arrow[urr, bend left, "f"] \arrow[drr, bend right] & Z_{n-1}(Q_\cdot) \times_{Z_{n-1}(N_\cdot)} N_n \arrow[ur] \arrow[dr] & & Z_{n-1}(N_\cdot) \\
& & Z_{n-1}(Q_\cdot) \arrow[ur] &
\end{tikzcd}$$

Thus, we have a factorization of f. By the lemma above, if we can show that h can factor degree wise as a monomorphism with projective cokernel followed by a fibration, the proof will be completed. We prove this by induction. Assume for $0 \leq k \leq n-1$ there exists $Q_k \in R-mod$ with the map $\partial : Q_k \to Q_{k-1}$ such that $\partial^2 = 0$, the chain maps $i : M_k \to Q_k$, $p : Q_k \to N_k$ such that $pi = f$ and i is injective with projective cokernel and the map $Q_k \to Z_{k-1}(Q_.) \times_{Z_{k-1}(N_.)} N_k$ is an epimorphism. Let $T_n = Z_{n-1}(Q_.) \times_{Z_{n-1}(N_.)} N_n$. We know by the proof of **MC5** (i) that we can factor h as

$$M_n \xrightarrow{i} M_n \oplus P(T_.)_k \xrightarrow{p} T_n$$

Setting $Q_n = M_n \oplus P(T_\bullet)_n$ we have completed the induction step and hence found our desired factorization. $\square$

We have now completed the proof in entirety.

3 Homotopy Category

We will define the devices needed to construct the homotopy category by defining a generalization of a homotopy from the category of topological spaces, **Top**, by using the machinery granted by the model structure. Then we will construct the homotopy category and compare it to a purely theoretical definition which was introduced in the introduction. The theoretical definition is much simplier, but lacks the geometrical intuition that guides us in the prior construction and does not come with the devices that we would like to have. Moreover, the theoretical definition does not give any implication that it is a locally small category.

Let $\mathcal{M}$ be a model category.

3.1 Homotopies

To define a homotopy in a model category we will first have to define right homotopy using path objects and then define its dual, left homotopy using cylinder objects. All of which have nice geometrical interpretations and, surprisingly, even algebraic ones.

3.1.1 Right Homotopy

Definition 3.1.1. Given $Y \in ob(\mathcal{M})$, the *diagnol map* is the map $\Delta : Y \to Y \prod Y$ in the diagram

$$\begin{tikzcd}
 & & Y \\
Y \arrow[r, dotted, "\Delta"] \arrow[urr, bend left, "id_Y"] \arrow[drr, bend right, "id_Y"'] & Y \prod Y \arrow[ur, "\pi_0"] \arrow[dr, "\pi_1"'] & \\
 & & Y
\end{tikzcd}$$

guaranteed by the universal property of coproducts.

Definition 3.1.2. [DS95] A *path object* of $Y \in ob(\mathcal{M})$ is any object P_Y such that there is a commutative diagram

$$\begin{tikzcd} Y \arrow[r, "\sim", "i"'] \arrow[rr, bend left, "\Delta"] & P_Y \arrow[r, "p"'] & Y \coprod Y \end{tikzcd}$$

where i is a weak equivalence. A path object P_Y is a *good path object* if p is a fibration and a *very good path object* if p is a fibration and i is a cofibration.

Remark 3.1.1. The path object is by no means unique nor does it have to be the path space of some object as one might guess. However, the path space in **Top** is in fact a path object as we will now see.

Example 3.1.1.

1. Let **Top** be the model category with the second model structure and $Y \in ob$**Top**. Then the path space, Y^I, is a path object in **Top**. This can be seen by the commutative diagram

$$\begin{tikzcd} Y \arrow[r, "\sim", "i"'] \arrow[rr, bend left, "\Delta"] & Y^I \arrow[r, "p"'] & Y \coprod Y \end{tikzcd}$$

   where i is the map that sends each point to the constant path at that point and p is the map that sends each path to its end points. This obviously factors the diagnol map.

2. Let $\mathbf{Ch}_{\geq 0}(R)$ be the model category given and $M_. \in \mathbf{Ch}_{\geq 0}(R)$. Then by the proof of **MC5** (i), the object $M_. \oplus P(M_. \coprod M_.)$ is a very good path object in $\mathbf{Ch}_{\geq 0}(R)$.

Definition 3.1.3. The maps $f, g : X \to Y$ are right homotopic, $f \simeq_r g$, if for some path object P_Y of Y, there exists a map $H : X \to P_Y$ such that the diagram

$$\begin{tikzcd} & P_Y \arrow[d, "p"] \\ X \arrow[ur, dotted, "H"] \arrow[r, "f \coprod g"] & Y \coprod Y \end{tikzcd}$$

commutes. The map H is said to be a right homotopy from f to g. If $H : X \to P_Y$ is a right homotopy and P_Y is a *(very) good path object*, then the map H is a *(very) good right homotopy.*

Lemma 3.1.1. [DS95] If Y is fibrant and P_Y is a good path object for Y, then the maps $\pi_0 \circ p,\ \pi_1 \circ p : P_Y \to Y$, where $\pi_0,\ \pi_1 : Y \coprod Y \to Y$ are the natural projections, are acyclic fibrations.

Proof. Since P_Y is a good path object, we have the commutative diagram

$$\begin{tikzcd} & & & Y \\ Y \arrow[r, "\sim", "i"'] \arrow[urrr, bend left, "id_Y"] \arrow[drrr, bend right, "id_Y"'] & P_Y \arrow[r, two heads, "p"'] & Y\prod Y \arrow[ur, "\pi_0"] \arrow[dr, "\pi_1"] & \\ & & & Y \end{tikzcd}$$

Since the identity maps on Y are obviously weak equivalences, $\pi_0 \circ p,\ \pi_1 \circ p$ are weak equivalences by **MC2**. Since $Y\prod Y$ is defined by the diagram

$$\begin{tikzcd} Y\prod Y \arrow[r, "\pi_0"] \arrow[d, "\pi_1"'] & Y \arrow[d, "p_0"] \\ Y \arrow[r, "p_1"] & * \end{tikzcd}$$

and Y is fibrant, $\pi_0,\ \pi_1$ are fibrations by 2.2.3. Since p is a fibration, $\pi_0 \circ p,\ \pi_1 \circ p$ are fibrations by composition. Hence, $\pi_0 \circ p,\ \pi_1 \circ p$ are acyclic fibrations. ☐

Lemma 3.1.2. [DS95] If $f \simeq_r g : X \to Y$, then there exists a good right homotopy from f to g. If in addition X is cofibrant, then there exists a very good right homotopy from f to g.

Proof. Since $f \simeq_r g : X \to Y$ for some path object P_Y, there exists a right homotopy $H : X \to P_Y$. By **MC5**, there exists a factorization of p such that the diagram

$$\begin{tikzcd} & P_Y \arrow[r, hook, "\sim", "i'"'] \arrow[d, "p"] & P'_Y \arrow[dl, two heads, "p'"] \\ X \arrow[ur, dotted, "H"] \arrow[r, "f\prod g"] & Y\prod Y & \end{tikzcd} \tag{3}$$

commutes. Thus, we have the diagram

$$\begin{tikzcd} Y \arrow[r, "\sim", "i"'] \arrow[rrr, bend left, "\Delta"] & P_Y \arrow[r, hook, "\sim", "i'"'] & P'_Y \arrow[r, two heads, "p'"'] & Y\prod Y \end{tikzcd} \tag{4}$$

where $i' \circ i$ is a weak equivalence by composition and p' is a fibration. Thus, P'_Y is a good cylinder object. Moreover, since the diagram (1) commutes, $i' \circ H$ is the required right homotopy. Hence, there exists a good right homotopy from f to g.

As for the second part, suppose X is cofibrant. By the first part, there exists a good path homotopy H between f and g with a good path object P_Y. Thus, by **MC5**, we have the commutative diagram

$$\begin{tikzcd}
Y \arrow[r, hook, "\sim", "i"'] \arrow[rr, bend left, "\Delta"] \arrow[dr, hook, "\sim", "i'"'] & P_Y \arrow[r, two heads, "p"] & Y \prod Y \\
& P'_Y \arrow[u, two heads, "p'"'] &
\end{tikzcd}$$

Since $p \circ p'$ is a fibration by composition, P'_Y is a very good path object for Y. Furthermore, p' is acyclic by **MC2**. Since X is cofibrant and the diagram

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d, hook] & P'_Y \arrow[d, two heads, "\sim"', "p'"] \\
X \arrow[r, "H"] & P_Y
\end{tikzcd}$$

commutes, there exists a lift $H' : X \to P'_Y$ by **MC4**. Thus, the diagram

$$\begin{tikzcd}
& P'_Y \arrow[d, "p'"] \\
& P_Y \arrow[d, "p"] \\
X \arrow[uur, "H'"] \arrow[ur, "H"] \arrow[r, "f \prod g"] & Y \prod Y
\end{tikzcd}$$

commutes and H' is the required right homotopy. Hence, there exists a very good right homotopy. $\square$

Lemma 3.1.3. [DS95] The relation $\simeq_r$ is an equivalence relation on $hom_{\mathcal{M}}(X, Y)$ if Y is fibrant.

3.1.2 Left Homotopy

Now, we define the dual of path object and right homotopy.

Definition 3.1.4. Given $X \in ob(\mathcal{M})$, the *folding map* is the map $\nabla : X \coprod X \to X$ in the diagram

$$\begin{tikzcd}
X \arrow[dr, "j_0"] \arrow[drr, bend left, "id_X"] & & \\
& X \coprod X \arrow[r, dotted, "\nabla"] & X \\
X \arrow[ur, "j_1"] \arrow[urr, bend right, "id_X"'] & &
\end{tikzcd}$$

given by the universal property of coproducts.

Definition 3.1.5. [DS95] A *cylinder object* of X is any object C_X such that there is a commutative diagram

$$\begin{tikzcd} X \coprod X \arrow[r, "i"'] \arrow[rr, bend left, "\nabla"] & C_X \arrow[r, "\sim", "p"'] & X \end{tikzcd}$$

where p is a weak equivalence. A cylinder object C_X is a *good cylinder object* if i is a cofibration and a *very good cylinder object* if i is a cofibration and p is a fibration.

Example 3.1.2. 1. Let **Top** be the model category with the second model structure and $X \in ob(\mathbf{Top})$. Then $X \coprod X = X \dot{\cup} X$ the disjoint union and the folding map maps both parts identically onto X. Since this map is obviously factored as

$$\begin{tikzcd} X \dot{\cup} X \arrow[r, "i"] \arrow[rr, bend left, "\nabla"] & X \times I \arrow[r, "p"] & X \end{tikzcd}$$

where I is the closed unit interval, i maps each X of the disjoint union to an end of $X \times I$, and p is the projection of $X \times I$ onto X, the geometrical cylinder $X \times I$ defines a cylinder object as it should. Also, note that p is a weak equivalence since it is a homotopy equivalence which implies it is a weak homotopy equivalence.

2. Let $\mathbf{Ch}_{\geq 0}(R)$ be the model category of chain complexes given above, $M_. \in \mathbf{Ch}_{\geq 0}(R)$ and $id_{M_.} : C_. \to C_.$ be the identity chain map. Then the homological mapping cylinder, $cyl(M_.)$ defined by $cyl(M_.)_n = M_n \oplus M_{n-1} \oplus M_n$ with boundary $\partial_n((m_0, m_1, m_2)) = (\partial m_0 + m_1, -\partial m_1, \partial m_2 + m_1)$, is a cylinder object of $M_.$. If we define $i : M_. \oplus M_. \to cyl(M_.)$ by $(m_0, m_1) \mapsto (m_0, 0, m_1)$ and $p : cyl(M_.) \to M_.$ by $(m_0, m_1, m_2) \mapsto m_0 + m_2$, then i, p are chain maps such that the diagram

$$\begin{tikzcd} X \oplus X \arrow[r, "i"'] \arrow[rr, bend left, "\nabla"] & cyl(M_.) \arrow[r, "p"'] & M_. \end{tikzcd}$$

commutes. Consider the map $q : M_. \to cyl(M_.)$ defined by $q(m) = (0, 0, m)$. Then obviously $pq = id_{M_.}$. A priori, pq is chain homotopic to $id_{M_.}$. Define a chain homotopy $\{s_n\}$ by $s((m_0, m_1, m_2)) = (0, m_0, 0)$. Then with a little calculation, we see that
$id((m_0, m_1, m_2)) - qp((m_0, m_1, m_3)) = \partial s + s\partial$. Thus, qp is chain homotopic to $id_{cyl(M_.)}$. Thus, the induced homomorphism p_* on homology groups is an isomorphism. Thus, p is a quasi-isomorphism. Hence, $cyl(M_.)$ is a cylinder object.

Definition 3.1.6. [GS06] The maps $f, g : X \to Y$ are left homotopic, $f \simeq_{\,l}\, g$, if for some cylinder object C_X of X, there exists a map $H : C_X \to Y$ such that the diagram

$$\begin{tikzcd} X \coprod X \arrow[r, "f \coprod g"] \arrow[d, "i"'] & Y \\ C_X \arrow[ur, dotted, "H"'] & \end{tikzcd}$$

commutes. The map H is said to be a left homotopy from f to g. If $H : C_X \to Y$ is a left homotopy and C_Y is a *(very) good cylinder object* then the map H is a *(very) good left homotopy.*

Example 3.1.3. 1. Let $X,\ Y \in$ **Top** and $f,\ g : X \to Y$. Then $f \simeq g$ in the classical sense if and only if there exists a map $F : X \times I \to Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$ which is equivalent to saying the diagram

$$\begin{tikzcd} X \arrow[d, "i_0"] \arrow[dr, "f"] & \\ X \times I \arrow[r, "F"] & Y \\ X \arrow[u, "i_1"] \arrow[ur, "g"'] & \end{tikzcd} \tag{5}$$

commutes. This is equivalent to the definition that $f \simeq_{\,l}\, g$.

2. A very similar argument shows that the classical notion of homotopy in Homological algebra is equivalent to left homotopy. The only difference is, you must show this degree wise and make sure that everything coummutes.

The following lemmas follow directly from the proofs for right homotopies.

Lemma 3.1.4. [DS95] If X is cofibrant and C_X is a good cylinder object for X, then the maps $i \circ j_0,\ i \circ j_1 : X \to C_X$, where $j_0,\ j_1$ are the natural inclusions, are acyclic fibrations.

Lemma 3.1.5. [DS95] If $f \simeq_l g : X \to Y$, then there exists a good left homotopy from f to g. If in addition X is fibrant, then there exists a very good left homotopy from f to g.

Lemma 3.1.6. [DS95] The relation $\simeq_{\,l}$ is an equivalence relation on $hom_{\mathcal{M}}(X, Y)$ if X is cofibrant.

3.1.3 Homotopy

Definition 3.1.7. The maps $f, g : X \to Y$ are homotopic, $f \simeq g$ if $f \simeq_{\,l}\, g$ and $f \simeq_{\,r}\, g$.

Lemma 3.1.7. [DS95] Let $f, g : X \to Y$.
- If Y is fibrant and $f \simeq_{\,r}\, g$, then $f \simeq_{\,l}\, g$.
- If X is cofibrant and $f \simeq_{\,l}\, g$, then $f \simeq_{\,r}\, g$.

Proof. We prove the first claim and the second follows by duality. Since $f \simeq_r g$, there exists a good right homotopy $H : X \to P_Y$ where P_Y is a good path object for Y (Lemma 3.1.2) such that the diagram

$$\begin{tikzcd} Y \arrow[r, "\sim", "i"'] \arrow[rr, bend left, "\Delta"] & P_Y \arrow[r, two heads, "p"'] & Y \prod Y \end{tikzcd}$$

commutes. Since Y is fibrant and P_Y is a good path object, $\pi_0 \circ p$ is an acyclic fibration where $\pi_0,\ \pi_1 : Y \prod Y \to Y$ are the natural projections (Lemma 3.1.1). By using **MC2** and **MC5**, we can find a good cylinder object for X such that the diagram

$$\begin{tikzcd} X \coprod X \arrow[r, hook, "i'"'] \arrow[rr, bend left, "\nabla"] & C_X \arrow[r, "\sim", "p'"'] & X \end{tikzcd}$$

commutes. Since the diagram

$$\begin{tikzcd} X \coprod X \arrow[r, "H \coprod (i \circ f)"] \arrow[d, hook, "i'"'] & P_Y \arrow[d, two heads, "\sim"', "\pi_0 \circ p"] \\ C_X \arrow[r, "f \circ p'"] & Y \end{tikzcd}$$

commutes, there exists a lift $h : C_X \to P_Y$. Moreover, $\pi_0 \circ p \circ h$ is the required left homotopy by the uniqueness of the universal property of products. $\square$

Remark 3.1.2. Given the maps $f, g : X \to Y$ where X is cofibrant and Y is fibrant, the equivalence classes of $\simeq_r$ and $\simeq_l$ concur.

3.2 Ho($\mathcal{M}$)

3.2.1 Constructive Homotopy Category

By now, it may have become apparent that objects that are fibrant and cofibrant have very nice properties. With the next theorem we will see they are precisely the objects we want to focus on to construct the homotopy category.

Theorem 3.2.1. Given a map $f : X \to Y$ such that $X,\ \ Y$ are both fibrant and cofibrant, then f is a weak equivalence if and only if f is a homotopy inverse.

Definition 3.2.1. For every object $X \in ob(\mathcal{M})$, the *fibrant replacement* of X is the object RX in the diagram

$$\begin{tikzcd} X \arrow[rr] \arrow[dr, hook, "\sim"] & & * \\ & RX \arrow[ur, two heads] & \end{tikzcd}$$

guaranteed by the functorial factorization $(\gamma,\ \delta)$ of **MC5**.
Similarly, for every object $X \in ob(\mathcal{M})$, the *cofibrant replacement* of X is the object QX in the diagram

$$\begin{tikzcd} \emptyset \arrow[rr, hook] \arrow[dr, hook] & & X \\ & QX \arrow[ur, two heads, "\sim"'] & \end{tikzcd}$$

guaranteed by the functorial factorization $(\alpha,\ \beta)$ of **MC5**.

Example 3.2.1. [GS06] It is worth noting that in $Ch_{\geq 0}(R)$, if $S^0(M)$ is the chain complex that has the R-module M in degree zero and the trivial R-module in all other degrees, then the cofibrant replacement of $S^0(M)$, $Q(S^0(M))$ given by

$$\begin{tikzcd} 0. \arrow[rr, hook] \arrow[dr, hook, "i"'] & & S^0(M) \\ & Q(S^0(M)) \arrow[ur, two heads, "\sim", "p"'] & \end{tikzcd}$$

is a projective resolution of M. To see this, note $Q(S^0(M))$ is of the form

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow 0 \longrightarrow \cdots$$

where $P_i \in R$-Mod. Since cofibrations in $\mathbf{Ch}_{\geq 0}(R)$ are injective chain maps with projective cokernel and the cokernel of i is $Q(S^0(M))$, P_i is projective for all $i \geq 0$. Since $Q(S^0(M))$ is quasi-isomorphic to $S^0(M)$, $H_n(P_n) \cong H_n(0) = 0$ for all $n > 0$. So $QM[0]$ is exact for $n > 0$. Moreover, since $M = H_0(S^0(M)) \cong H_0(Q(S^0(M))) = P_0/B_0(P_1)$, $\mathrm{coker}(P_1 \to P_0) = P_0/Im(P_1) = M$. Hence, $Q(S^0(M)) \to M$ is a projective resolution.

Lemma 3.2.1. [DS95] For every map $f : X \to Y$ there exists a map $f^* : QRX \to QRY$ such that f is a weak equivalence if and only if f^* is a weak equivalence. The map f^* is unique up to homotopy.

Let $\mathcal{M}_{cf}$ be the category with all objects from $\mathcal{M}$ that are both fibrant and cofibrant and the morphism sets the same as the morphism sets in $\mathcal{M}$. Let $\mathcal{M}_{cf}/\simeq$ be the category with the same objects as $\mathcal{M}_{cf}$ but let the morphism sets be the quotient of the morphism sets in $\mathcal{M}$ by homotopy.

Theorem 3.2.2. The fibrant-cofibrant replacement map $QR : \mathcal{M} \to \mathcal{M}_{cf}/\simeq$ definied by $X \mapsto QRX$ for every $X \in ob(\mathcal{M})$ and for every $f \in hom_{\mathcal{M}}(X,\ Y)$ $f \mapsto [f^*] \in hom_{\mathcal{M}_{cf}}(QRX,\ QRY)$ is a functor.

Definition 3.2.2. Given a model category $\mathcal{M}$ the homotopy category of $\mathcal{M}$ is the category $Ho(\mathcal{M})$ where

$$ob(Ho(\mathcal{M})) = ob(\mathcal{M})$$

and

$$hom_{Ho(\mathcal{M})}(X, Y) = hom_{\mathcal{M}}(QRX, QRY)/\simeq .$$

Theorem 3.2.3. Let $H_{\mathcal{M}} : \mathcal{M} \to Ho(\mathcal{M})$ be defined by $X \mapsto X$ for all $X \in ob(M)$ and $f \mapsto QR(f)$ for all $f \in Mor(M)$. Then $H_{\mathcal{M}}$ is a functor. Furthermore, $H(f)$ is an isomorphism if and only if f is a weak equivalence.

3.2.2 Non-constructive Homotopy Category

Now we formally define the theoretical definition of the homotopy category that was introduced in the introduction. To do this, we will now define a localization of a category with respect to a specific class of morphisms.

Let $\mathscr{C}$ be a category and $\mathscr{W} \subset Mor(\mathscr{C})$.

Definition 3.2.3. [KS06] A localization of $\mathscr{C}$ with respect to $\mathscr{W}$ is the data of a big category $\mathscr{W}^{-1}\mathscr{C}$ and a functor $F : \mathscr{C} \to \mathscr{W}^{-1}\mathscr{C}$ satisfying:

1. $F(w)$ is an isomorphism for all $w \in \mathscr{W}$,
2. for any big category $\mathscr{D}$ and any funtor $G : \mathscr{C} \to \mathscr{D}$ such that $G(w)$ is an isomorphism for all $w \in \mathscr{D}$, there exists a functor $U : \mathscr{W}^{-1}\mathscr{C} \to \mathscr{D}$ such that the diagram

$$\begin{tikzcd} \mathscr{C} \arrow[r, "G"] \arrow[d, "F"] & \mathscr{D} \\ \mathscr{W}^{-1}\mathscr{C} \arrow[ur, "U"] & \end{tikzcd}$$

   commutes up to isomorphism,
3. if U_1, U_2 are two objects of $\mathscr{W}^{-1}\mathscr{C}^{\mathscr{D}}$ then the natural map

$$hom_{\mathscr{D}^{\mathscr{W}^{-1}\mathscr{C}}}(U_1, U_2) \to hom_{\mathscr{D}^{\mathscr{C}}}(U_1 \circ F, U_2 \circ F)$$

   is bijective.

Now, the theoretical definition of the homotopy category in the introduction is simply the localization of the category with respect to the weak equivalences i.e. $\mathscr{W} =$ the class of weak equivalences of $\mathscr{C}$.

Example 3.2.2. As an example in Homological Algebra, the localization of $\mathbf{Ch}(R)$ with respect to the class of quasi-isomorphisms is the derived category $\mathbf{D}(R)$.

3.2.3 Equivalence

For model categories to fulfill their purpose, the homotopy category constructed from a model category must be isomorphic to the localization of the model category with respect to the class of weak equivalences. With a quick result about the functor $H_{\mathcal{M}}$ discussed above we will see that this is in fact true.

Lemma 3.2.2. [DS95] Given $f \in Mor(\mathcal{M})$, f is a weak equivalence in $\mathcal{M}$ if and only if $H(f)$ is an isomorphism in $Ho(\mathcal{M})$.

Theorem 3.2.4. [DS95] The functor $H_{\mathcal{M}}$ given above is a localization of $\mathcal{M}$ with respect to the class of weak equivalences $\mathscr{W}$.

Thus, by the universal property of localizations, $Ho(\mathcal{M}) \cong \mathscr{W}^{-1}\mathcal{M}$ where $\mathscr{W}$ is the class of weak equivalences.

Example 3.2.3. Since the derived category $\mathbf{D}(R)$ of chain complexes over R is a localization of $\mathbf{CH}(R)$ with respect to the class of quasi-isomorphisms, $\mathbf{D}(R) \cong \mathrm{Ho}(\mathbf{Ch}(R))$.

4 Morphisms

Now that we have the desired categories in which to work, we would like to find the appropriate morphisms between them. These morphisms follow immediately after a little excursion into the construction of left and right derived functors. As their name may suggest, they root from the subject of Homological Algebra as we will see. After a discussion of their existence, we wil see how they lead directly to the definition of our morphisms of model categories which we will call Quillen Functors.

4.1 Derived Functors

Definition 4.1.1. Let $\mathscr{C}$ be a model category, $F : \mathscr{C} \to \mathscr{D}$ be a functor and $H_{\mathscr{C}} : \mathscr{C} \to \mathrm{Ho}(\mathscr{C})$ be the natural functor of $\mathscr{C}$ into its homotopy category. Then a *left derived functor* of F is a pair $(LF,\, l)$ such that the diagram

$$\begin{tikzcd} \mathscr{C} \arrow[rr, "F"] \arrow[dr, "H_{\mathscr{C}}"'] & \arrow[d, phantom, "\Uparrow l"] & \mathscr{D} \\ & \mathrm{Ho}(\mathscr{C}) \arrow[ur, "LF"'] & \end{tikzcd}$$

commutes and if $(G,\, l')$ is any other such pair, there exists a natural transformation $t : G \to LF$ such that $l \circ (t \circ id_{H_{\mathscr{C}}}) = l'$ where here $id_{H_{\mathscr{C}}}$ is taken to be the identity natural transformation on H.
Similarly, a *right derived functor* of F is a pair $(RF,\, r)$ such that the diagram

$$\begin{tikzcd} \mathscr{C} \arrow[rr, "F"] \arrow[dr, "H_{\mathscr{C}}"'] & \arrow[d, phantom, "\Downarrow r"] & \mathscr{D} \\ & \mathrm{Ho}(\mathscr{C}) \arrow[ur, "RF"'] & \end{tikzcd}$$

commutes and if $(G,\, r')$ is any other such pair, there exists a natural transformation $t : RF \to G$ such that $(t \circ id_{H_{\mathscr{C}}}) \circ r = r'$.

Remark 4.1.1. [Hir03] As dealing with all the compositions of natural transformations may seem difficult, Hirschhorn gives a plausible figurative understanding of the universal properties of left and right derived functors. The left derived functor is a functor that is the closest to F on the left and the right derived functor is the closest functor to F on the right.

Remark 4.1.2. It is also worth noting that the universal properties imply that a left or right derived functor are unique up to unique isomorphism. Thus, from this point foward we will refer to the left and right derived functor.

After defining left and right derived funtors, we naturally lead to a discussion of total left and total right derived functors, as they are a particularly important case of left and right derived functors, respectively.

Definition 4.1.2. Let $\mathscr{C}$, $\mathscr{D}$ be model categories and $F : \mathscr{C} \to \mathscr{D}$ a functor. Then the *total left derived functor* $(\mathbf{L}F, \mathbf{l})$ is simply the left derived functor of the composition $H_{\mathscr{D}} \circ F : \mathscr{C} \to \mathrm{Ho}(\mathscr{D})$. Similarly, the *total right derived functor* $(\mathbf{R}F, \mathbf{r})$ is simply the right derived functor of the composition $H_{\mathscr{D}} \circ F$.

If we think of the derived category as the analog of the homotopy category, then one might begin to see the relevance of the terminology since the purpose of the total left and right derived functors in Homological Algebra are to extend a functor to derived categories. In our case, we are extending a functor between model categories to the their respecitve homotopy categories. Now, our mission is to find sufficient conditions for the left and right derived functor to exist. In order to do this, we will first prove two lemmas.

Lemma 4.1.1. [Hir03] Let $\mathscr{C}$ be a model category and $F : \mathscr{C} \to \mathscr{D}$ be a functor.

1. Let X, Y be cofibrant objects in $\mathscr{C}$ and the map $f : X \to Y$ be a weak equivalence. Then f factors as

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, hook, "\sim", "i"'] & & Y \\ & X' \arrow[ur, two heads, "\sim"', "p"] & \end{tikzcd}$$

   and there exists an acyclic cofibration $q : Y \to X'$ such that $pq = id_Y$.

2. Let X, Y be fibrant objects in $\mathscr{C}$ and the map $f : X \to Y$ be a weak equivalence. Then f factors as

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, hook, "\sim", "i"'] & & Y \\ & X' \arrow[ur, two heads, "\sim"', "p"] & \end{tikzcd}$$

   and there exists an acyclic fibration $q : X' \to X$ such that $qi = id_X$.

Proof. We will prove the first part and the second follows by duality. Since X, Y are cofibrant, we have the commutative diagram

$$\begin{tikzcd} \emptyset \arrow[r, hook] \arrow[d, hook] & X \arrow[d, "j_0"] \\ Y \arrow[r, "j_1"] & X \coprod Y \end{tikzcd}$$

Thus, j_0, j_1 are acyclic cofibrations by proposition 2.2.3. By MC5, we have the factorization

$$\begin{tikzcd} X \coprod Y \arrow[rr, "g \coprod id_Y"] \arrow[dr, hook, "k"'] & & Y \\ & Z \arrow[ur, two heads, "\sim", "l"'] & \end{tikzcd}$$

Since cofibrations are closed under composition, $k \circ j_0,\ k \circ j_1$ are cofibrations. Since $g,\ l,\ id_Y$ are weak equivalences and $g = l \circ (k \circ j_0),\ id_Y = l \circ (k \circ j_1)$, $k \circ j_0,\ k \circ j_1$ are weak equivalences by MC2. Hence, letting $i = k \circ j_0,\ p = l,\ q = l \circ (k \circ j_1)$ and $X' = Z$, we have the desired claim. $\square$

Corollary 4.1.1. [Hir03] Let $\mathscr{C}$ be a model category and $F : \mathscr{C} \to \mathscr{D}$ be a functor.

1. If F maps acyclic cofibrations between cofibrant objects in $\mathscr{C}$ to isomorphisms, then F maps weak equivalences between cofibrant objects in $\mathscr{C}$ to isomorphisms.

2. If F maps acyclic fibrations between fibrant objects in $\mathscr{C}$ to isomorphisms, then F maps weak equivalences between fibrant objects in $\mathscr{C}$ to isomorphisms.

Proof. Let $X,\ Y$ be cofibrant objects and $f : X \to Y$ be a weak equivalence. Then f factors as

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, hook, "\sim", "i"'] & & Y \\ & X' \arrow[ur, two heads, "\sim", "p"'] & \end{tikzcd}$$

and there exists an acyclic fibration $q : X' \to X$ such that $qi = id_X$ by the lemma above. Since $i,\ q$ are acyclic cofibrations and X' is cofibrant, $F(i),\ F(q)$ are isomorphisms. Hence, $F(f) = F(q)^{-1}Fi$ is an isomorphism. $\square$

Theorem 4.1.1. [Hir03] Let $\mathscr{C}$ be a model category and $F : \mathscr{C} \to \mathscr{D}$ a functor.

1. If F maps acyclic cofibrations between cofibrant objects to isomorphisms in $\mathscr{D}$, then the left derived functor (LF, s) of F exists. Moreover, if X is cofibrant, s_X is an isomorphism.

2. If F maps acyclic fibrations between fibrant objects to isomorphisms in $\mathscr{D}$, then the right derived functor (RF, s) of F exists. Moreover, if X is fibrant, s_X is an isomorphism.

Proof. Let $D : \mathscr{C} \to \mathscr{D}$ be defined by $D(X) = F(QX)$ and $D(f) = F(Q(f))$ where Q is the cofibrant replacement functor and $f \in hom_{\mathscr{C}}(X,\ Y)$. Since D is a composition of functors, D is a functor. If $f : X \to Y$ is a weak equivalence in $\mathscr{C}$, then $Q(f)$ is a weak equivalence between cofibrant objects. Thus, there is a unique functor $LF : \mathrm{Ho}(\mathscr{C}) \to \mathscr{D}$ by the universal property of localizations. Notice how we conveniently denoted this functor by LF. Define a natural transformation $s : LF \circ H_{\mathscr{C}} \to F$ by $s(X) = F(i_X)$ where i_X is the natural weak equivalence between QX and X. Since $F(i_X) : F(QX) \to F(X)$ and $F(QX) = D(X) = LF \circ H_{\mathscr{C}}$, $s(X)$ is in fact a natural transformation from $LF \circ H_{\mathscr{C}}$ to F. Now, suppose $(G,\ s')$ is a similar pair such that $G : \mathrm{Ho}(\mathscr{C}) \to \mathscr{D}$ and $s' : G \circ H_{\mathscr{C}} \to F$. We need to find a natural transformation $t : G \circ H_{\mathscr{C}} \to LF \circ H_{\mathscr{C}}$ such that $s \circ (t \circ id_{H_{\mathscr{C}}}) = s'$. Since s' is a natural transformation, $F(QX) = LF \circ H_{\mathscr{C}}$ and $F(i_X) = s(X)$, we have the commutative diagram

$$\begin{tikzcd}
(G \circ H_{\mathscr{C}})(QX) \arrow[r, "s'(QX)"] \arrow[d, "(G\circ H_{\mathscr{C}})(i_X)"'] & (LF \circ H_{\mathscr{C}})(X) \arrow[d, "s(X)"] \\
(G \circ H_{\mathscr{C}})(X) \arrow[r, "s'(X)"'] & F(X)
\end{tikzcd}$$

Since i_X is a weak equivalence, $(G \circ H_{\mathscr{C}})(i_X)$ is an isomorphism. Thus, let $t = s'(QX) \circ ((G \circ H_{\mathscr{C}})(i_X))^{-1}$. Since i_X is an acyclic cofibration, by hypothesis $F(i_X)$ is an isomorphism. Thus t must be unique. □

Corollary 4.1.2. Let $\mathscr{C}, \mathscr{D}$ be model categories and $F : \mathscr{C} \to \mathscr{D}$ a functor.

1. If $H_{\mathscr{D}} \circ F$ maps acyclic cofibrations between cofibrant objects to isomorphisms in $\mathrm{Ho}(\mathscr{D})$, then the total left derived functor $(\mathbf{L}F, s)$ of F exists.

2. If $H_{\mathscr{D}} \circ F$ maps acyclic fibrations between fibrant objects to isomorphisms in $\mathrm{Ho}(\mathscr{D})$, then the total right derived functor $(\mathbf{R}F, s)$ of F exists.

To further the relevance between the left and right derived functors of model categories and the left and right derived functors in Homological Algebra we apply our new terminology to the tensor functor.

Example 4.1.1. Let $\mathbf{Ch}(R)$ and $\mathbf{Ch}(\mathbb{Z})$ have the usual model structures and $M \in mod - R$. Then we have the functor

$$\mathbf{Ch}(R) \xrightarrow{M \otimes_R -} \mathbf{Ch}(\mathbb{Z}) \xrightarrow{H} \mathrm{Ho}(\mathbf{Ch}(\mathbb{Z}))$$

We first show that there exists the total left derived functor $\mathbf{L}(H \circ M \otimes_R -)$. By the corollary 4.1.2, we need only show that $H \circ M \otimes_R -$ maps acyclic cofibrations between cofibrant objects to isomorphisms in $\mathrm{Ho}(\mathbf{Ch}(\mathbb{Z}))$. Suppose $i : A_. \to B_.$

is an acyclic cofibration in $\mathbf{Ch}(R)$. Since i is injective degreewise, we have the short exact sequence

$$0 \longrightarrow A_. \longrightarrow B_. \longrightarrow B/A \longrightarrow 0$$

Thus, we have a long exact sequence of homology groups. Since i is a quasi-isomorphism, B/A is acyclic. Also, since i is injective with projective cokernel for $n \geq 0$, $(B/A)_n$ is projective for all $n \geq$. By lemma 2.5.2, $Z_n(B/A)$ is projective and

$$B/A \cong \oplus_{n\geq 1} D^n(Z_{n-1}(B/A)).$$

Thus,

$$B \cong A \oplus B/A \cong A \oplus (\oplus_{n\geq 1} D^n(Z_{n-1}(B/A))).$$

Since $M \otimes_R -$ commutes with direct sums,

$$(M \otimes_R -)(B) \cong (M \otimes_R -)(A) \oplus (\oplus_{n\geq 1}(M \otimes_R -)D^n(Z_{n-1}(B/A))).$$

Since $(M \otimes_R -)D^n(Z_{n-1}(B/A))$ is acyclic and homology commutes with direct sum, $H((M \otimes_R -)(B)) \cong H((M \otimes_R -)(A))$. Thus, $(M \otimes_R -)(i)$ is a weak equivalence. Since H maps weak equivalences to isomorphisms, $H \circ (M \otimes_R -)(i)$ is an isomorphism. Hence, $\mathbf{L}(H \circ M \otimes_R -)$ exists.
Since the cofibrant replacement of $S^0(N)$ is a projective resolution $P_.$ of N and $S^0(N)$ is weakly equivalent to $P_.$, we have

$$\mathbf{L}(M \otimes_R -)(S^0(N)) \cong \mathbf{L}(M \otimes_R -)(P_.) \cong M \otimes_R P_.$$

Thus,

$$H_i(\mathbf{L}(M \otimes_R -)(S^0(N)) \cong H_i(M \otimes_R P_.) = Tor_i^R(M, N)$$

Theorem 4.1.2. [DS95] Let $\mathscr{C}, \mathscr{D}$ be model categories and

$$\mathscr{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{D}$$

be an adjoint pair. That is, F is a left adjoint to G. If F preserves cofibrations and G preserves fibrations, then

$$\mathrm{Ho}(\mathscr{C}) \underset{\mathbf{R}G}{\overset{\mathbf{L}F}{\rightleftarrows}} \mathrm{Ho}(\mathscr{D})$$

are adjoints. Moreover, if for every cofibrant object $X \in \mathrm{ob}(\mathscr{C})$ and every fibrant object $Y \in \mathrm{ob}(\mathscr{D})$, $F(X) \to Y$ is a weak equivalence if and only if its adjoint morphism $X \to G(Y)$ is a weak equivalence, then $\mathbf{L}F$ and $\mathbf{R}G$ are inverse equivalences of categories.

4.2 Quillen Functors

Since the weak equivalences in a model category are precisely the isomorphisms in the homotopy category, it is quite easy to see that the best choice of morphisms between model categories would be precisely the ones that hold this structure. Moreover, these morphisms should certainly preserve constructions dependent on the model category structure such as cylinder objects, path objects, and homotopies. Furthermore, an "isomorphism" should be a functor on model categories that induces an equivalence of homotopy categories. As of which, theorem 4.1.2 gives a complete description of such functors which we will now formally define.

Definition 4.2.1. [GS06] Let $\mathscr{C}$, $\mathscr{D}$ be model categories and

$$\mathscr{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{D}$$

be an adjoint pair. Then F (resp. G) is a *left (resp. right) Quillen functor* if F (resp. G) preserves cofibrations (resp. fibrations) and weak equivalences between cofibrant (resp. fibrant) objects. The pair (F, G) is called a *Quillen pair.*

Lemma 4.2.1. [Hir03] Let $\mathscr{C}, \mathscr{D}$ be model categories and

$$\mathscr{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{D}$$

be an Quillen pair.

1. If X is a cofibrant object of $\mathscr{C}$ and CX is a cylinder object of X, then $F(CX)$ is a cylinder object for FX.

2. If Y is a fibrant object of $\mathscr{D}$ and PY is a path object of Y, then $G(PY)$ is a path object for GY.

Lemma 4.2.2. [Hir03] Let $\mathscr{C}, \mathscr{D}$ be model categories and

$$\mathscr{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{D}$$

be an Quillen pair.

1. If $f, g : X \to Y$ are left homotopic maps in $\mathscr{C}$, then $F(f)$ and $F(g)$ are left homotopic in $\mathscr{D}$.

2. If $f, g : X \to Y$ are right homotopic maps in $\mathscr{D}$, then $G(f)$ and $G(g)$ are right homotopic in $\mathscr{C}$.

Theorem 4.2.1. [Hir03] Let $\mathscr{C}, \mathscr{D}$ be model categories and

$$\mathscr{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{D}$$

be an Quillen pair. If X is a cofibrant object of $\mathscr{C}$ and Y is a fibrant object of $\mathscr{D}$, then the isomorphism

$$hom_{\mathscr{D}}(FX, Y) \cong hom_{\mathscr{C}}(X, GY)$$

induces an isomorphism

$$hom_{\mathscr{D}}(FX, Y)/\simeq \;\; \cong \;\; hom_{\mathscr{C}}(X, GY)/\simeq .$$

Now, we give the definition of the functors that give what we would like "isomorphisms" to be of model categories. Again, this definition follows from theorem 4.1.2.

Definition 4.2.2. [Hir03] Let $\mathscr{C}, \mathscr{D}$ be model categories and

$$\mathscr{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{D}$$

be an Quillen pair. Then F (G) is a *left (right) Quillen equivalence* if for every cofibrant object $X \in \mathrm{ob}(\mathscr{C})$ and every fibrant object $Y \in \mathrm{ob}(\mathscr{D})$ $F(X) \to Y$ is a weak equivalence if and only if its adjoint morphism $X \to G(Y)$ is a weak equivalence. The pair (F, G) is called a *Quillen equivalence*.

As the next theorem is truely a restatement of theorem 4.1.2, we state it anyhow for definiteness.

Theorem 4.2.2. [Hir03] Let $\mathscr{C}, \mathscr{D}$ be model categories and

$$\mathscr{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{D}$$

be an Quillen pair. If (F, G) is a pair of Quillen equivalences, then the induced adjoint pair

$$\mathrm{Ho}(\mathscr{C}) \underset{\mathbf{R}G}{\overset{\mathbf{L}F}{\rightleftarrows}} \mathrm{Ho}(\mathscr{D})$$

form an equivalence of homotopy categories.

Example 4.2.1. [GS06] Let $f \in hom_{CRings}(R, S)$ and res_f be the restriction of scalars functor. Then

$$\mathbf{Ch}(R) \underset{res_f}{\overset{S\otimes_R -}{\rightleftarrows}} \mathbf{Ch}(S)$$

is a Quillen pair. Moreover, if $R = S$, then this is a Quillen equivalence.

Example 4.2.2. [GS06] Let $|-|$ be the geometric realization functor and $S(-)$ be the singular set functor. Then

$$s\mathbf{Sets} \underset{S(-)}{\overset{|-|}{\rightleftarrows}} \mathbf{CGH}$$

is a Quillen equivalence where **CGH** is the category of compactly generated weak Hausdorff spaces.

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