

Computing the Endogenous Mortgage Rate without Iterations

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Abstract

A number of mortgage prepayment models require a specification of the mortgage rate process. Usually some ad-hoc models are used (e.g., a Treasury yield plus some constant). Recently, a number of papers appeared where authors utilized a mortgage rate implied by the current yield curve (so called endogenous mortgage rate). However, the existing computational algorithms suffer from the curse of dimensionality and, consequently, are problematic to use for full scale problems. A computational algorithm, proposed in this paper, does not require iterations. Moreover, the algorithm is tractable in the sense that its complexity is equivalent to the problem of mortgage valuation. The numerical example is based on a PDE computation. An implementation of a Monte Carlo method is also discussed.

1 Introduction

A fixed-rate level-payment mortgage at first glance looks like a fixed-coupon bond. However, the cash flow of this mortgage is not certain, as its name may imply. Borrowers might prepay mortgages due to various reasons (in this case the lender “loses” anticipated interest payments¹). This uncertainty (prepayment option) makes mortgage-based securities riskier than traditional fixed-coupon bonds.

Prepayment models try to predict the borrowers’ prepayment behavior based on information available to the market. There is still no agreement on the “correct” prepayment model. The major trend on

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¹This might be bad for the mortgage investor (if those payments are higher than the current interest rates can provide) or good (if the situation is the opposite).

Wall Street is to model the prepayment behavior empirically. For example, the prepayment rates are regressed against some explanatory variables, and the 10-year Treasury yield is often considered as one of the most influential predictors.

If we ask “why 10-year Treasury?” then the most common answer would be that this rate closely tracks the 30-year fixed-rate mortgage rate. So, in fact, the empirical models use some benchmark for the mortgage rate. This idea was implemented in the mortgage-rate based (or MRB for short, see [4]) approach to prepayment modeling. It assumes that the borrowers’ refinancing decision is based mainly on the comparison of the *contract* and *current* (available for refinancing) mortgage rates (the internet is abundant with “calculators” which based on this comparison, tell one how much one “saves” if he/she refinances). If this approach to model refinancing incentive is taken, then the investor needs to model the *mortgage rate* to be able to model the prepayment process in order to price or hedge mortgage-backed securities. This mortgage rate model needs to be in agreement with the mortgage rates implied by the resulting prepayment behavior and the underlying interest rate model (we call such mortgage rate models *endogenous*).

In spite of its importance, the problem was not considered in depth in academia until recently. The first general endogenous mortgage rate model under sub-optimal prepayment assumption was developed by Goncharov in [4] and [3], where the endogenous mortgage rate is formulated as a fixed-point of a functional operator. Pliska [9] investigated the problem in a discrete time setting. Goncharov, Okten, and Shah [6] applied randomized quasi Monte Carlo method for the mortgage rate computation. Bhattacharjee and Hayre [1] presented a mortgage rate model (called MOATS model) which in fact is a version of the iteration procedure of the fixed-point mortgage rate problem in Goncharov [4], [3] given a certain “educated” initial guess. This guess comes from solving an auxiliary mortgage problem over a $[30, 60]$ -year interval where mortgages originated at time $t \in [30, 60]$ have maturity of $60 - t$ years.² Additionally, the MOATS model actually computes mortgage rate for interest-only mortgages, not the standard (most popular) 30-year fixed-rate fixed-payment fully amortized mortgage.

In this paper we consider the problem of computing the mortgage rate process under the MRB prepayment assumption (i.e., the refinancing decision is governed by comparison of the original and current mortgage rates). In [4], the problem of the endogenous mortgage rate was formulated as a fixed point of some functional operator. In [3] and

²In general, if the interest rate tree in [1] was propagated to $30 \times (n + 1)$ years (instead of $60 = 30 \times 2$ years). The question of how close the MOATS mortgage rate approximates the endogenous mortgage rate is addressed in Goncharov [2].

[6] an iterative algorithm was proposed for computation of the endogenous mortgage rate process. This algorithm computes the mortgage rates with the help of iterations on every point on some mesh over the interest rate domain. The new algorithm in this paper is based on finding “level curves” of this operator and approximating the mortgage rate function with an interpolation of these level curves. With a certain condition on the mortgage rate grid, the algorithm does not require iterations. Finding a level curve requires computation of only two conditional expectations, and the number of level curves equals the number of points on the mortgage rate grid, which is one-dimensional. This is in sharp contrast to the existing algorithm proposed in [3] and [6] where the number of expectations to be computed is the number of points needed to reconstruct the mortgage rate function (surface) over the domain of state factors and is a number growing exponentially with the dimensionality of the problem (the complexity of the Citigroup’s MOATS methodology [1] is of the same order).

Let us emphasize that the algorithm can be applied not only to an MRB specification of the prepayment, but to any other prepayment model. Other models might not have a dependence on the mortgage rate process (as the option-based approach in [7], etc.) and, therefore, computing the endogenous mortgage rate for a specific initial state is a matter of solving a *scalar* non-linear equation (which is a simple problem given that the valuation method with this model is developed). However, if one wants to compute the mortgage rate as a function of “inputs” (e.g., to know/forecast how the mortgage rate will change) the level curve idea in this paper can simplify the numerical computations.

The numerical results in this paper confirms the previous findings on the endogenous mortgage rate in [5] and [6]. The endogenous mortgage rate function exhibits a highly nonlinear behavior around “common” values of the interest rate. This peculiar behavior is strikingly different from the behavior of empirically modeled mortgage rates. The mortgage rates are generally higher if a substantial refinancing risk is present. The “jump” represents an interest rate region which separates “low”- and “high”-refinancing risk regions. The MRB assumption (i.e., the assumption that the refinancing is driven by comparison of the mortgage rates) makes these regions “accented.” This peculiarity of the endogenous mortgage rate function needs further investigation since it affects prepayment modeling and pricing (and hedging) of associated mortgage securities. The point of future empirical studies might be to answer the question: “what if instead of a benchmark we use an endogenous mortgage rate?” A more accurate model (as opposed to using some long rate as a mortgage rate benchmark) would reduce risks to investors and, consequently, might lower mortgage rates in general.

The rest of the paper is organized as follows. In Section 2 we give a general model specification which we consider throughout the paper.

In Section 3 we present the computational algorithm in the case of a one-factor interest rate model. In Section 3.1 we present numerical results and in Section 3.2 we study several interesting properties of the endogenous mortgage rates computed in Section 3.1. We discuss the endogenous mortgage rate in the case of heterogenous borrowers in Section 4. In Section 5 we discuss a multi-factor implementation of our algorithm. In particular, we will see that the complexity of the algorithm is equivalent to the complexity of the fundamental mortgage valuation problem. This is in strike contrast to the general non-tractability of known algorithms which are based on finding mortgage rates given the state space values. In Section 6 we extend the algorithm to the case when the transaction costs are “small” or absent. In this case the procedure requires iterations, but stays “tractable” in the sense that it keeps the complexity of the mortgage valuation problem. We shortly discuss the implementation of Monte Carlo method in Section 7.

2 The Model

2.1 Set-up

We consider the following contract: a borrower takes a loan of P_0 dollars at the origination and assumes the obligation to pay scheduled coupons at fixed rate c continuously for duration T of the contract. The loan is secured by collateral on some specified real estate property, which obliges the borrower to make the payments. The borrower has the right to settle his/her obligation and prepay the outstanding principal in a lump sum. The interest on the principal is compounded according to a contract mortgage rate m^t , which is determined at the origination time t and is fixed for the duration of the contract. Given that the mortgage is fully amortized (i.e., the loan should be completely payed off after T years), the outstanding principal for time $s \in [t, t + T]$ can be easily computed and is given by the function $P(s - t, m^t) = P_0 \frac{1 - e^{m^t(T - (s - t))}}{1 - e^{m^t T}}$ (where $s - t$ is the time passed after the origination).

We assume that at any time t it is possible to invest one unit in a default-free deposit account at a short-term interest rate r_t and to “roll-over” the proceeds until a later time s for a market value at that time of $e^{\int_t^s r_\theta d\theta}$. This interest rate is modeled as a deterministic (measurable) function $r(X_t)$ of some time-homogenous³ Markov process (state variable) $X_t = (X_t^1, \dots, X_t^p)$ with the support $D \subset \mathbb{R}^p$, i.e., for any $X_0 \in D$ we have $X_t \in D$ a.s. for any $t > 0$. In what follows, we require

³We might include the non-time-homogenous case by considering, for example, $X_t^1 = t$.

the monotonicity of the function $r(x)$ with respect to the components of x . In general, this requirement is desirable but not necessary. It will make the computation more efficient by allowing to avoid iterations. If one cannot reformulate the problem to satisfy this monotonicity assumption then the method does not change conceptually but might require iterations.

We consider the usual information structure described by a natural filtration of the state process $(\Omega, \mathbf{F}, \{\mathcal{F}\}_{t \geq 0}, Q)$. We interpret $\{\mathcal{F}_t\}_{t \geq 0}$ as a model of the flow of public information which is not borrower-specific. The filtration is an intrinsic feature of the market: this means that all traders have the same information available at any given time. The mortgage rate process m^t should be adapted to the filtration $\{\mathcal{F}_t\}_{t \geq 0}$.

2.2 Specification of prepayment

The prepayment time is modeled as the time of the first jump of a generalized Poisson process (the so-called Cox process). This means that the probability of prepayment is driven by some \mathcal{F}_t -intensity process γ_t . Intuitively speaking, given that a borrower did not prepay as of time t , the probability that he/she will prepay over the next “short” time interval Δt is $\gamma_t \Delta t$. Equivalently, given a “large” hypothetical pool of “homogeneous” borrowers, γ_t is the rate of prepayment (as a function of a state of economy \mathcal{F}_t) in terms of proportion (of the borrowers staying in this pool at time t) per unit of time. Given this intuition, the prepayment intensity γ_t can be regarded as the prepayment rate.

As we pointed out in the introduction, prepayment modeling is a very complicated problem. Modeling the prepayment rate as a function of a set of observable predictors, we essentially specify the probability of prepayment of some given borrower in a given time interval conditioned on some given economy state. Clearly, this probability depends on many factors. It can be the contract and current mortgage rates (the borrower compares them to judge how profitable refinancing would be), interest rate yield curve (e.g., if the market expect the interest rate to decrease, then borrowers might be inclined to postpone refinancing), so called media effect (historically low mortgage rates get special attention in media and induce higher refinancing), borrower’s credit rating, education, location, loan-to-value ratio, and many many others. All such predictors, with the exception of mortgage rates, are assumed to be modeled by the state process (7), i.e., they are assumed to be exogenously specified. Thus, for a borrower with a mortgage originated at time t , his/her prepayment intensity (“prepayment rate”) at time $s \geq t$ is assumed to be a function of the state variable X_s (which includes all the information about exogenously specified factors such as interest rates, borrower’s credit, etc.), contract m^t and the current

mortgage rate m^s , i.e., $\gamma_s = \gamma(X_s, m^t, m^s)$.

The prepayment decision in the case of $m^s > m^t$ (time t being here an origination time) is not, generally, financially based. Therefore, we forbid the refinancing to a mortgage with a higher mortgage rate.⁴ The prepayment might happen due to exogenous reasons only. Technically it means that the prepayment intensity does not depend on the current mortgage rate m^s in this case. Moreover, we assume that there are transaction costs associated with the refinancing which are measured in terms of mortgage rates. These conditions imply the following specification of the intensity:

$$\gamma_s := \begin{cases} \gamma_1(X_s, m^t, m^s) & m^s < m^t + \delta \\ \gamma_2(X_s, m^t) & m^s \geq m^t + \delta \end{cases} \quad (1)$$

where δ is the transaction cost in percents and for simplicity is taken as a constant. Note that $\gamma_2(X_s, m^t)$ has the interpretation of the intensity of prepayment due to exogenous reasons.⁵ To single out the dependence on the mortgage rates, for the rest of the paper we hide the dependence on X_s by using the notation $\gamma_s(m^t, m^s)$ for the intensity process defined in (1).

For the most part of the paper we assume that borrowers are homogenous. That is, the probability of prepayment given some state of economy (the current mortgage rate in our case) is the same across all borrowers. This is known to be not the case in practice and there is an empirical proof of this fact in the form of observed burn-out effects: the total prepayment rate in a pool of mortgages which already experienced a refinancing wave is lower (the pool is “burned-out”) because borrowers who are more likely to refinance did so in the time of the previous refinancing wave. We address the case of heterogenous borrowers in Section 4. The problem of the mortgage rate in the non-homogenous case is easier from the computational point of view because a solution in this case is smoother (as we will see) and, therefore, behaves better. From this point of view, our “simplification” (homogenous borrowers) is indeed only a notational one and can be regarded as an extreme (from a computational viewpoint) case.

⁴In fact, the prepayment intensity (not refinancing part) might be influenced by higher mortgage rates because borrowers might try to avoid to prepay (even if this decision is not financially based) if the current mortgage rate is very high. For example, a relocation decision due to a new job offer might be easier to make in the case if a new mortgage would have slightly higher mortgage rate than in the case of significantly higher current mortgage rate. Nevertheless, the prepayment intensity due to exogenous reasons is generally much smaller than the refinancing intensity and this kind of dependence of the prepayment intensity on the “negative” refinancing incentive is negligible. However, if desired, the extension of the algorithm to this case is possible with incorporation of iterations.

⁵Then the intensity of refinancing is $\gamma_1(X_s, m^t, m^s) - \gamma_2(X_s, m^t)$ if $m^s < m^t + \delta$ and zero otherwise.

2.3 Mortgage rate equation

It can be shown (see [3]) that the price of the mortgage M_t (originated at time t and having contract mortgage rate m^t) can be expressed as the following expectation:

$$M_t = P(0, m^t) + \mathbf{E} \left[\int_t^{t+T} (m^t - r_u) P(u - t, m^t) e^{-\int_s^u (\gamma_\theta + r_\theta) d\theta} du \mid X_s \right]. \quad (2)$$

To exclude an arbitrage opportunity, the price of the mortgage at the origination should be equal to the initial outstanding principal, i.e., $M_t = P(0, m^t) = P_0$. Using this fact in equation (2), we obtain

$$0 = \mathbf{E} \left[\int_t^{t+T} (m^t - r_u) P(u - t, m^t) e^{-\int_t^u (\gamma_\theta(m^t, m^\theta) + r_\theta) d\theta} du \mid X_t \right], \quad (3)$$

which can be written as

$$m^t = \frac{\mathbf{E} \left[\int_t^{t+T} r_u P(u - t, m^t) e^{-\int_t^u (\gamma_\theta(m^t, m^\theta) + r_\theta) d\theta} du \mid X_t \right]}{\mathbf{E} \left[\int_t^{t+T} P(u - t, m^t) e^{-\int_t^u (\gamma_\theta(m^t, m^\theta) + r_\theta) d\theta} du \mid X_t \right]}. \quad (4)$$

Thus, using the time-homogeneity of the state process X_t , the mortgage rate process can be defined as $m^t = m(X_t)$, where the function $m(x)$ is the solution of the following functional equation

$$m(x) = \frac{\mathbf{E} \left[\int_0^T r_u P(u, m(x)) e^{-\int_0^u (\gamma_\theta(m(x), m(X_\theta)) + r_\theta) d\theta} du \mid X_0 = x \right]}{\mathbf{E} \left[\int_0^T P(u, m(x)) e^{-\int_0^u (\gamma_\theta(m(x), m(X_\theta)) + r_\theta) d\theta} du \mid X_0 = x \right]}. \quad (5)$$

The purpose of this paper is to provide a computationally efficient algorithm for the computation of a function $m(x)$ which satisfies (5).

3 Solving the Model

Let us define the following operator:

$$\mathcal{A}[f(\cdot)](p, x) := \frac{\mathbf{E} \left[\int_0^T r_u P(u, p) e^{-\int_0^u (\gamma_\theta(p, f(X_\theta)) + r_\theta) d\theta} du \mid X_0 = x \right]}{\mathbf{E} \left[\int_0^T P(u, p) e^{-\int_0^u (\gamma_\theta(p, f(X_\theta)) + r_\theta) d\theta} du \mid X_0 = x \right]}, \quad (6)$$

where $p \in \mathbb{R}^+$, $x \in D \subset \mathbb{R}^n$, and $f : D \rightarrow \mathbb{R}^+$.

Given this definition, equation (5) can be written in an operator form as

$$m(x) = \mathcal{A}[m(\cdot)](m(x), x), \quad x \in D.$$

In principle, this equation can be solved with iterations (see [6]): given an iteration function $m_n(x)$, the following iteration is defined as a function

$$m_{n+1}(x) = \mathcal{A}[m_n(\cdot)](m_n(x), x).^6$$

But let us state an important point: the computation of $m_{n+1}(x)$ for even one fixed x requires the knowledge of the previous iteration function $m_n(x)$ for all $x \in D$. Therefore, the computation of the iteration function $m_{n+1}(x)$ requires computation of $\mathcal{A}[m_n(\cdot)](m_n(x), x)$ for all $x \in D$. In particular, for a fixed p the computation of the function $\mathcal{A}[m_n(\cdot)](p, x)$ requires computation of only *two* expectations, but the computation of the function $\mathcal{A}[m_n(\cdot)](m_n(x), x)$ requires evaluation of the separate expectations in (6) for each x (i.e., $m_n(x)$ -specific expectations). Practically it means that the number of expectations to be computed is twice the number of grid points on the domain D . The number of points required to approximate a surface (i.e., the mortgage rate function over D) grows exponentially with the dimension of the domain in general. If we multiply this complexity for one iteration by the number iterations required, then we can easily see that this iteration procedure is problematic to employ for real industrial applications.

An alternative computational approach is proposed in this paper. To illustrate our basic idea in the simplest case possible, in this section we assume that the only stochastic process in the model is the instantaneous interest rate r_t which is a *one-factor* diffusion process with $r_t \in D = (0, \infty)$. In this simple interest rate environment the whole

⁶A generalized (with the interest rate tree propagated to $30 \times (n + 1)$ years) version of MOATS mortgage rate model in [1] can be written as iterations $m_{n+1}(x) = \mathcal{A}[m_n(\cdot)](m_{n+1}(x), x)$, where $m_0(x)$ is taken as a solution to an auxiliary mortgage rate problem over a $[30 \times n, 30 \times (n + 1)]$ -year interval where mortgages originated at time $t \in [30 \times n, 30 \times (n + 1)]$ have maturity of $30 \times (n + 1) - t$ years.

yield curve is determined by the value of r_t , and thus the mortgage rate process m^t is merely a function of the interest rate r_t , i.e., $m^t = m(r_t)$.

We assume the monotonicity of the mortgage rate function, i.e., $m(r^1) \leq m(r^2)$ if $r^1 \leq r^2$. Together with the specification of the intensity (1), this implies that a borrower's prepayment intensity does not depend on the behavior of the mortgage rate function for values of the interest rate higher than r^* , where r^* is the value of the interest rate at the time when the borrower acquired his/her mortgage.

The main computational idea is somewhat similar to considering a Lebesgue instead of Riemann integration. Instead of trying to find the mortgage rate for a given interest rate (as we do in a general iteration idea), we fix a mortgage rate value and try to find an appropriate interest rate value.

Let us fix a mortgage rate grid $\{m_n = m_0 + n\Delta m\}_{n=0}^N$, where Δm is some value less than the transaction cost δ (depends on a precision desired), N is a number which gives a desired mortgage rate range, and m_0 is an infimum of the mortgage rate values. In the current set-up m_0 is a solution to the *scalar* equation $m_0 = \lim_{r \rightarrow 0^+} \mathcal{A}[1](m_0, r)$, where "1" stands for "no refinancing" condition.⁷

Now, for every m_n we find an appropriate interest rate value r_n in turn. Suppose r_k , $k = 0, \dots, n-1$, are known. Then we define $\tilde{m}_n(r)$, $r \leq r_{n-1}$ as some interpolation function on known pairs $\{(r_k, m_k)\}_{k=0}^{n-1}$. Given the choice of Δm and the monotonicity of the mortgage rate function, for any possible monotone extension $m(r)$ of $\tilde{m}_{n-1}(r)$ beyond r_{n-1} we have an a priori estimate $m(r) > m_n - \delta$ for all $r > r_{n-1}$. This implies an a priori fact that the prepayment intensity γ_t for computing r_n (that is finding at what interest rate value the mortgage with the rate m_n was originated) does not depend on the mortgage rate behavior for the interest rate values $r > r_{n-1}$. To formalize this fact, we define the following function

$$\bar{m}_{n-1}(r) := \begin{cases} \tilde{m}_{n-1}(r) & r \leq r_{n-1} \\ \infty & r \geq r_{n-1} \end{cases}$$

which merely extends $\tilde{m}_n(r)$ in such a way that the refinancing for $r > r_n$ is excluded.

Given these considerations, r_n is found by solving the equation

$$m_n = \mathcal{A}[\bar{m}_{n-1}](m_n, r)$$

for r . We can perform this procedure for $n = 1, \dots, N$, and the approximation of a solution to (5) is taken as $\tilde{m}_N(r)$.

⁷Computationally, $\lim_{r \rightarrow 0^+} \mathcal{A}[1](m_0, r)$ would be evaluated using the boundary values of the numerically estimated expectations.

Note that in this case the number of expectations to be computed is twice the number of the *mortgage rate* grid points. It is known (see [6]), that the mortgage rate function exhibits a highly non-linear behavior (the “jump”) around some interest rate value which is in the region of “common” interest rate values (i.e., the “jump” cannot be ignored). This “jump” separates the low and high refinance risk regions. Because of this phenomena, the iteration procedure in [3], [6] needed a fine interest grid or an adaptive grid to deal with the “jump.” Formally, applying the procedure considered in this paper, an adaptive interest grid is computed naturally as a by-product of the method⁸ (inverse values of the uniform mortgage rate grid will be more dense in the region of higher gradient of the mortgage rate function). Additionally, we do not need to do iterations at all!

More impressive advantages of this numerical procedure will be seen in a multi-factor setting (Section 5).

3.1 Numerical Example

In this section we explain the computation of the endogenous rate when the expectations are computed with PDEs. We consider a one-factor CIR interest rate model. Where the interest rate is given by a solution of the following stochastic differential equation

$$dr_t = \alpha(\mu - r_t)ds + \sigma\sqrt{r_t}dW_t, \quad r_t = r \in (0, \infty), \quad (7)$$

and where W_t is a Brownian motion. We consider the following parameters for the CIR interest rate model: $\alpha = 0.3$, $\mu = 0.07$, and $\sigma = 0.115$.

Let $U[m(\cdot)](m_o, x)$ and $W[m(\cdot)](m_o, x)$ be the numerator and denominator of the righthand side of the operator $\mathcal{A}[m(\cdot)](m_o, x)$ in definition (6). For the case we consider here we have $x = r$. For a given function $m(\cdot)$, under some technical conditions (see [3]), these functions are solutions of the following differential equations:

$$\frac{\partial U}{\partial t} + \mathcal{L}U - [r + \gamma(r, m_o, m(r))]U + rP(t, m_o) = 0, \quad (8)$$

$$\frac{\partial W}{\partial t} + \mathcal{L}W - [r + \gamma(r, m_o, m(r))]W + P(t, m_o) = 0, \quad (9)$$

$$U_T(r) = 0, \quad W_T(r) = 0, \quad (t, r) \in (0, T) \times D, \quad (10)$$

where operator \mathcal{L} is the generator of the CIR diffusion state process r_t , i.e.,

$$\mathcal{L} := \frac{\sigma^2 r}{2} \frac{\partial^2}{\partial r^2} + \alpha(\mu - r) \frac{\partial}{\partial r}. \quad (11)$$

⁸As we will see in Section 3.1, it is sufficient to have a very coarse mortgage rate grid to have this effect. In this case an adaptive mortgage rate grid might be useful. See the note in Section 3.1.

Therefore, the function $\mathcal{A}[m(\cdot)](m_o, x)$ is defined as a ratio of solutions of PDEs (8) and (9).

Next, we assume the following special case of the prepayment intensity function:

$$\gamma_t = \begin{cases} \gamma_1, & \text{if } m^0 > m^t + \delta \\ \gamma_0, & \text{if } m^0 \leq m^t + \delta, \end{cases} \quad (12)$$

where γ_0 , γ_1 , and δ are constants and represent the intensity of prepayment due to exogenous reasons, the intensity of prepayment in the case when it is financially justifiable, and the transaction costs of refinancing in terms of mortgage percentage.

We assume $\gamma_0 = 0$, $\gamma_1 = 0.65$, and $\delta = 1\%$. This choice implies that borrowers check their refinancing opportunities in stochastic time intervals with an average “waiting” time of $1/\gamma_1 \approx 1.5$ years. They require about 10% of decrease on their monthly mortgage payments (if we compute these “savings” implied by 1% mortgage rate decrease) to refinance and they do not prepay for exogenous reasons. This specification is somewhat similar to the specification of the prepayment in [10], where the prepayment rate, however, was assumed to be governed by the borrower’s liability (the so called option-based approach).

The values for the prepayment and interest rate parameters are rounded values of the equivalent parameters estimated in [10] with the only exception being δ , since in [10] the transaction costs are expressed in percents of the outstanding principal while in our case it is expressed in terms of percents of the mortgage rate.

We use a linear interpolation for $\tilde{m}_n(r)$. The result of the computation of the endogenous mortgage rate with a 10-step uniform mortgage rate grid (from m_o to 10%) is shown on Fig. (1). As we can see, the algorithm produces a solution (the dashed line) which tracks closely a “continuous” solution (solid line) computed with iterations (which are computed for *every* point on the interest grid). The exception is the “jump” region where the deviation from the “continuous” solution is due to the linear interpolation.

In Fig. 2 we magnify the region of the “jump” and show the mortgage rate function computed with our algorithm with 5 (dash-dotted line), 10 (dashed line), and 20 (dotted line) steps of the uniform mortgage rate grid. The appropriate horizontal lines (dash-dotted, dashed, or dotted according the number of steps used) shows the levels of the mortgage rates used to compute the corresponding interest rates. The mortgage rate solution computed with 40 step mortgage rate grid is visually indistinguishable from the “continuous” solution.

Fig. 3 shows the absolute value of the deviation of the level set solutions from the “continuous” mortgage rate function. All grids produce the mortgage rate functions which are practically indistinguishable from the “continuous” solution for the small interest values (a half

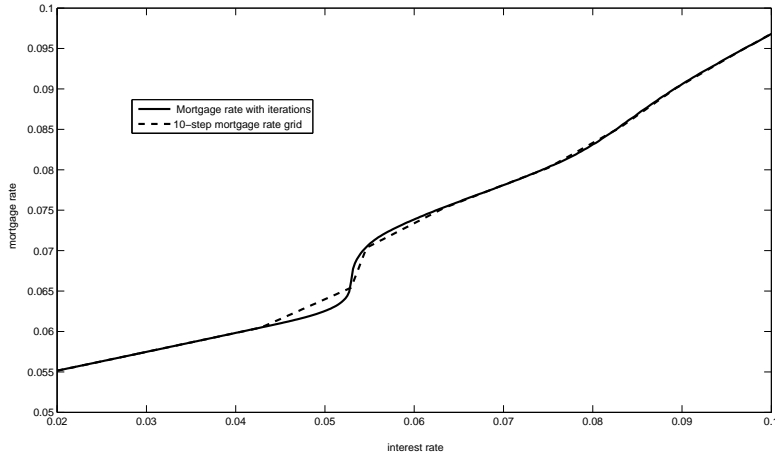


Figure 1: Comparison of the mortgage rate function computed with iterations at every point of the interest rate grid (solid line) and the mortgage rate function computed with 10 steps mortgage rate grid (dashed line).

of a percentage point for 5 step grid and less than a quarter of a point for other grids for the interest rate values less than 4%). We can see that large deviations for 5 and 10 step mortgage rate grids are due to points of linear interpolations. For 20 step grid the errors are less than 2.5 basis points except for the neighborhood of the “jump.” For the 40 step grid the errors are around of one basis point. The only exclusion is a narrow neighborhood (plus/minus 10 basis points of the “jump” location) of the “jump” where the high gradient makes the deviation to be around 10 basis points.

Note on adaptive grid. As we see in Fig. 2 and Fig. 3, the error is negligible for the interest rate values to the left of the “jump.” Around the “jump” the linear interpolation is not adequate for large Δm , though the values of the mortgage rates at points which correspond to the mortgage rate level values (m^n for r^n in our notation) were very close in either case (less than a half of a point even in 5 or 10 step case). This suggests the following idea for an adaptive mesh. Assume the mortgage rate function for values lower than m^n is found (i.e., $m(r)$ is approximated for $r \leq r^n$). We take a large mortgage rate step, say, $\Delta m_{n+1} = \delta$ and compute the interest rate r^{n+1} for the level $m^n + \Delta m$. Then we take a half step, i.e., we compute the interest rate $r^{n+1/2}$ for the level $m^n + \Delta m/2$. Now if $m^n + \Delta m/2$ is within some tolerance from $\tilde{m}_{n+1}(r^{n+1/2})$ we proceed with $\tilde{m}_{n+1}(r)$ as an accepted

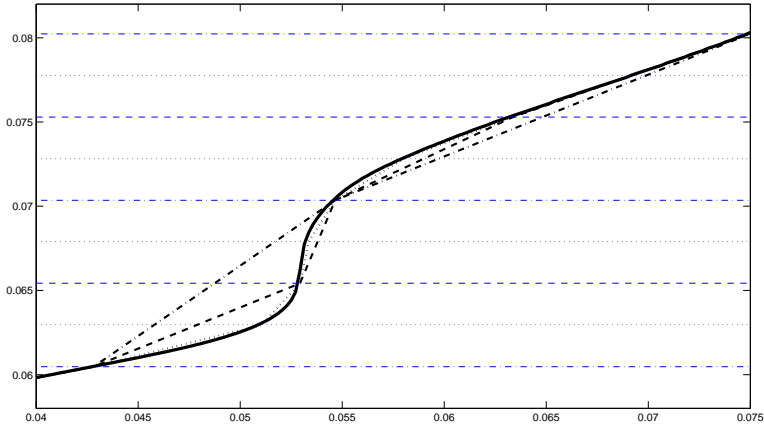


Figure 2: Level sets for the algorithm with 5, 10, and 20 step mortgage rate grids.

approximation of the mortgage rate function for the interest rate values $r \leq r^{n+1}$, otherwise we divide the intervals $(m_n, m_n + \Delta m/2)$ and $(m_n + \Delta m/2, m_n + \Delta m)$ further in half and check if $r^{n+1/4}$ and $r^{n+3/4}$ are satisfactorily close to the interpolation with the “inserted” point $r^{n+1/2}$, etc.

3.2 On some qualitative properties.

As we pointed out, if the mortgage rate process is needed for certain purposes, then a common practice is to take some long-term Treasury yield as a benchmark for the mortgage rate (e.g., the mortgage rate process is modeled as the 10-year Treasury yield plus some exogenously specified constant). If we look at the endogenous mortgage rate (Fig. 1) then we can see that its non-linear behavior (e.g., the “jump” around 5.5% and the “hump” around 9%) does not allow a satisfactory uniform fit for any Treasury yield. Let us take a Treasury yield as a function of the initial short interest rate (which uniquely specifies the current yield curve in our one-factor CIR interest rate model setting) and pick a term which “fits” the endogenous mortgage rate only on some interval of the interest rate values. As we can see on Fig. 4, we can separate three regions: 1) on the left to the “jump” (the interest rates less than around 5.5%) the 30-year Treasury fits the mortgage rate; 2) the 10-year Treasury is close to the mortgage rates between the “jump” and the “hump” (the interest rates between around 5.5% and 8%); 3) the

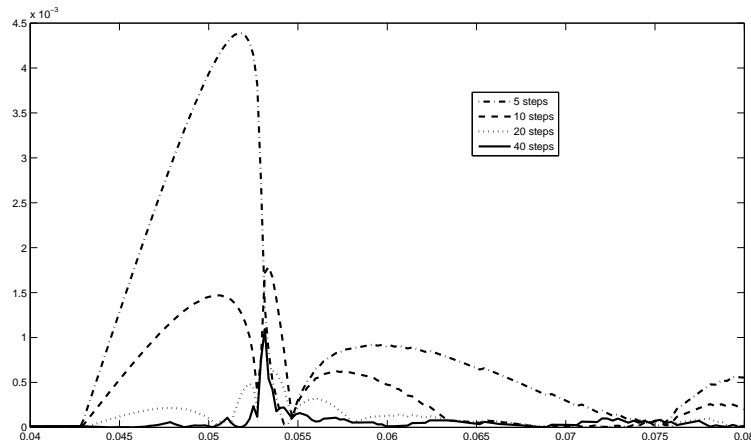


Figure 3: Deviations of the solutions based on the 5, 10, 20, and 40 step grids from the mortgage rate function computed with iterations for every point on the interest rate grid.

5-year Treasury gives a good fit after the “hump” (the interest rates larger than around 8%). On Fig. 4, we added appropriate constants to 5- and 10-year Treasury yields to get the “fit”.

This simple experiment illustrates why 10-year Treasury yield might work nicely in certain circumstances. Let us emphasize that this fit is a purely ad-hoc approach and in different economic situations different terms might be required to be used as benchmarks to have a good performance. Additionally, any such benchmark ignores the presence of the “jump.”⁹

Let us see what mortgage rate we get if the prepayment model is based on the 10-year Treasury yield benchmark (i.e., one version of a simple empirical prepayment model). Assume that the refinancing decision is triggered by the 10-year Treasury yield, i.e., the prepayment rate is specified as¹⁰

$$\gamma_t = \begin{cases} \gamma_1, & \text{if } y_{10}(r^0) > y_{10}(r^t) + \delta \\ \gamma_0, & \text{if } y_{10}(r^0) \leq y_{10}(r^t) + \delta \end{cases} \quad (13)$$

⁹As we will show in the next section, the “jump” is smoothed out in the more realistic case of heterogenous borrowers. However, the transaction between the lower and higher interest rate region is still present and cannot be ignored.

¹⁰Compare with the definition of the intensity (12) where the refinancing decision is driven by the endogenous mortgage rate.

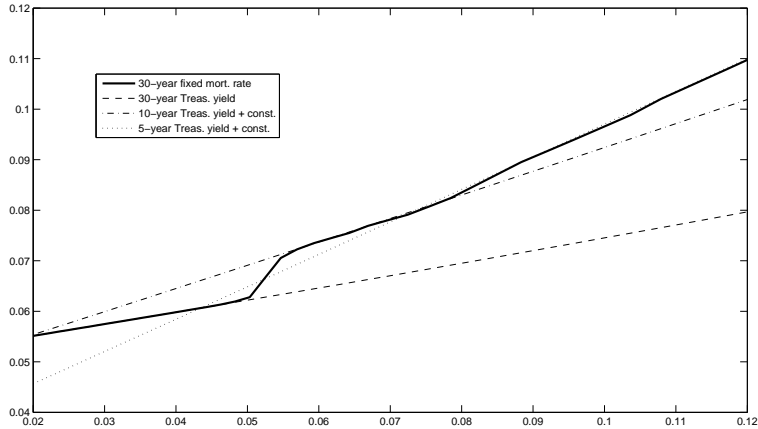


Figure 4: Comparison of the mortgage rates and certain Treasury yields as functions of the short interest rate.

where $y_{10}(r)$ is the 10-year Treasury yield given that the current interest rate is r . Fig. 5 shows the comparison of the mortgage rates implied by this empirical model (dashed line) and the endogenous mortgage rates (solid line). The mortgage rate implied by the 10-year Treasury yield grossly overestimates the endogenous mortgage rate for the interest rates to the left of the jump.

An explanation of the significantly lower endogenous values is that the lenders giving the lower mortgage rates for borrowers would be compensated by the virtual absence of the refinancing in lower rate situations in the presence of the 1% transaction costs. However, the risk of refinancing is significant if the mortgage rates are given using the 10-year Treasury benchmark.

This compensation (i.e., reduction of the refinancing risk) is not enough for the interest rates higher than around 5.5%. Between 5.5% and 6.5% the 10-year Treasury benchmark implies lower (of about 10 basis points) mortgage rates. In this case the empirical model underestimates the refinancing risk (which is presented by the “jump”). Between 7% and 8.5% the 10-year Treasury benchmark implies higher (about 10 basis points) mortgage rates. In this case the refinancing risk predicted by the endogenous mortgage rate process is lower because of the higher endogenous mortgage rates in the (5.5%, 6.5%) interest rate region and, consequently, the lower refinancing incentive $m(r_0) - m(r_t)$ there.

Another important point we would like to make here is that the

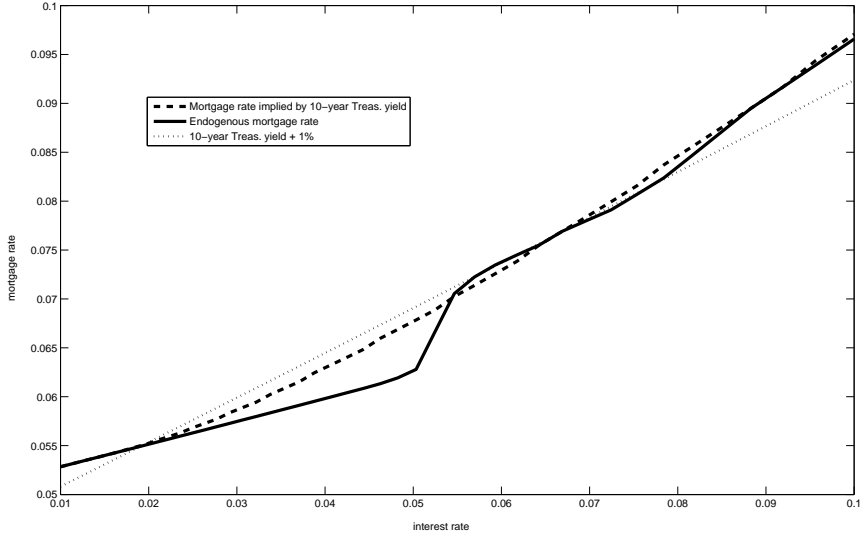


Figure 5: Comparison of the endogenous mortgage rates with the 10-year Treasury yield benchmark and the mortgage rates implied by the prepayment model which is based on this benchmark.

mortgage rate implied by the 10-year Treasury benchmark is *not* a solution to the mortgage rate equation (5). Compare the mortgage rate implied by this benchmark with the 10-year Treasury yield itself. In Fig. 4 we “attempt” to fit the 10-year Treasury to the implied mortgage rate by adding 1% to the yield. The slope of the implied mortgage rate function is about the same for interest rates between 3% and 5% but significantly higher for higher interest rates. That is, the prepayment models which use the 10-year Treasury benchmark for the mortgage rate modeling are not arbitrage free.

4 Heterogenous Borrowers

In this section we illustrate the extension of our framework to the case of pools with heterogenous borrowers. Assume the set of all possible borrower’s intensity functions is given by the parameterized family $\{\gamma_t^\omega\}_{\omega \in \Omega}$. Next, we assume that we have statistically estimated the distribution Φ_ω of borrowers in some geographical region where the mortgage rate model are to be used. Then the distribution of borrowers in a pool which is just originated in that geographical region can be

viewed as an unbiased estimator of Φ_ω . The deviation of the actual borrowers distribution in the pool from Φ_ω represents the idiosyncratic risk for investors and should not be priced. Taking this into account and repeating the manipulations of section 2.3, we conclude that the mortgage rate is given by the equation

$$m(x) = \frac{\int_{\Omega} \mathbf{E} \left[\int_0^T r_u P(u, m(x)) e^{-\int_0^u (\gamma_\theta^\omega(m(x), m(X_\theta)) + r_\theta) d\theta} du \mid X_t = x \right] d\Phi_\omega}{\int_{\Omega} \mathbf{E} \left[\int_0^T P(u, m(x)) e^{-\int_0^u (\gamma_\theta^\omega(m(x), m(X_\theta)) + r_\theta) d\theta} du \mid X_t = x \right] d\Phi_\omega}. \quad (14)$$

The computation of the Φ_ω -integral might be evaluated with the help of some quadrature if Φ_ω is a sufficiently smooth distribution.

In Fig. 6 we illustrate a solution to this equation assuming that all the borrowers can be divided into three (with equal proportions) groups: 0.7% transaction costs, 1% transaction costs, and 1.3% transaction costs. The other parameters are taken to be the same as in Section 3.1.

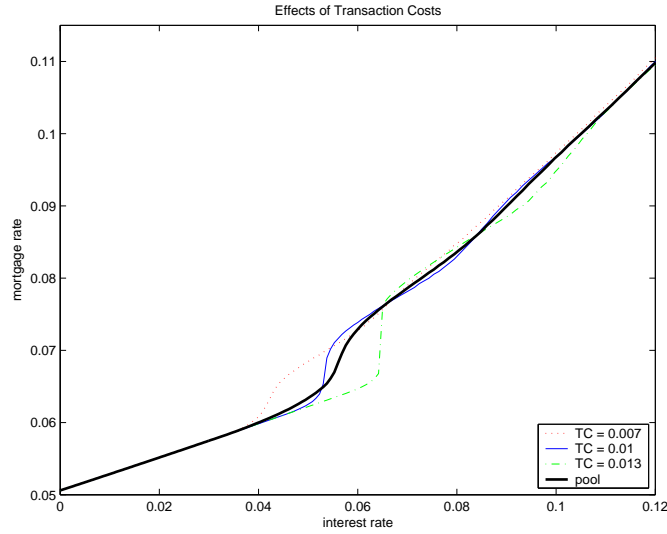


Figure 6: The mortgage rate in the case of heterogenous borrowers.

As we can see, in the presence of heterogenous borrowers the “jump” region is smoothed out. From the computational point of view it means that the interpolation between level curves is going to be closer in this (more realistic!) case. In particular, the step size of the mortgage rate

mesh Δm might be taken larger (without losing the precision) than in the case of homogenous borrowers.

5 Multi-factor Extension

As we pointed out, iteration procedures, which are based on finding the mortgage rates for given interest rate values, suffer from a curse of dimensionality. Let us emphasize that this is a problem of the procedure itself and not of the expectations computation. Additionally, procedures of “consecutive” iterations in [3] and [6] is problematic to extend to multi-dimensional models because a space of dimension greater than one is not ordered (see [6]). But with the procedure considered in this paper, the extension to higher dimensions is natural. Moreover, the number of the expectations to be computed in this algorithm does not grow with the dimensionality of the state process. This is a serious advantage over iteration procedures where we have an exponential growth (with respect to the state process dimension) in the number of expectation computations.

Assume that the interest rate process $r_t = r(X_t)$ is a function of the p -dimensional state process $X_t \in R^n$. We discretize the (one-dimensional!) values of the mortgage rates m^n , $n = 0, \dots, N$, the same way we did in the one-factor case.¹¹ Now, consider the equation

$$m_1 = \mathcal{A}[1](m_1, x),$$

where $x \in R^p$ and the constant function 1 stands for “no refinancing” condition. The solution of this equation is a manifold l_1 of $\mathcal{A}[1](m_1, x)$ as function of x . We will refer to this solution as a “level curve”, i.e., as if $p = 2$. Given the monotonicity of the mortgage rate function $m(x)$ with respect to the components of x , this level curve separates the region (in terms of x) of higher and lower (than m_1) mortgage rates. Let us use notation L_1 for the lower mortgage rate domain, i.e., $L_1 = \{x \mid m(x) < m_1\}$. The borrowers who receive their mortgages in the situation when the yield curve implies that the state variable is from this domain will never refinance their mortgages because for any $X_t \in L_1$ we have $m(X_t) > m_1 - \delta$ (according to the way the mortgage rate grid is defined) and for any $X_t \in D \setminus L_1$ we have $m(X_t) > m_1$. The approximation of the mortgage rate over L_1 can be defined as an interpolation between m_0 and m_1 on l_1 .

Before we proceed, let us formulate the major idea. A given level curve l_n (being defined below) defines the boundary of the domain of dependence L_n for the mortgages originated on this level curve (the mortgage rates are higher “outside” of l_n , i.e., where $X_t \in D \setminus L_n$, and

¹¹That is m_0 is the infimum of the mortgage rates, the grid is uniform, and $\Delta m \leq \delta$.

do not influence the prepayment decision). Therefore, the operator \mathcal{A} for finding l_{n+1} (i.e., for mortgages originated on l_{n+1}) will depend only on domain L_n , where the mortgage rate function can be approximated by an interpolation on the level curves l_k , $k = 0, \dots, n$. Therefore, finding the level curves l_n consecutively for $n = 1, \dots, N$, we construct an approximation of the mortgage rate function by an interpolation on these level curves.

Let us assume that the level the curves l_k for $k = 1, \dots, n - 1$ are known. Similarly to the one-factor case, we define $\tilde{m}_{n-1}(x)$ as an interpolation between the known level curves l_k . Next, we define

$$\bar{m}_{n-1}(x) := \begin{cases} \tilde{m}_{n-1}(x) & x \in L_{n-1} \\ \infty & x \notin L_{n-1} \end{cases}$$

Then the level curve l_n is defined as a solution to the following equation

$$m_n = \mathcal{A}[\bar{m}_{n-1}](m_n, x). \quad (15)$$

As we see, in the multi-factor case the number of equations to be solved is the same as in the one-factor case (since in either case we discretize the *one-dimensional* mortgage rate space). The growth of complexity comes from the computation of expectations and finding level curves only. The problem of finding a level curve is computationally cheaper than evaluating the conditional expectations in the definition of the operator \mathcal{A} itself. At the same time, the computation of these expectations has equivalent complexity to evaluating the fundamental problem of computing the mortgage price (compare the mortgage price equation (2) to the definition (6) of the operator \mathcal{A}). It means that if one implements an empirical multi-factor mortgage model, then the computational complexity of including an endogenous mortgage rate instead of its benchmark can be practically implemented too.

6 Extension to zero transaction cost

If the transaction cost δ is small, then the mortgage rate grid might be unnecessarily fine because of our requirement $\Delta m \leq \delta$. If we assume zero transaction cost, than the procedure does not work the way it was described ($\Delta m = 0$ does not make sense). But the algorithm can be extended to overcome this complication.

Let us assume we are given the level curves l_k , $k = 1, \dots, n - 1$, and want to define the level curve l_n , i.e., to find values of the interest rates which imply the mortgage rate m_n . First, we take an initial guess l_n^0 such that $l_n^0 \notin L_{n-1}$ (otherwise, the mortgage rate surface is not monotone). For example, a good initial guess might be some

extrapolation of previous level curves $\{l_k\}_1^{n-1}$. Then we define $\tilde{m}_n^0(x)$ as an interpolation between the known level curves l_k plus our initial guess l_n^0 . Next, define

$$\bar{m}_n^0(x) := \begin{cases} \tilde{m}_n^0(x) & x \in L_n^0 \\ \infty & x \notin L_n^0 \end{cases}$$

where L_n^0 is the region “inside” l_n^0 . To find l_n we use iterations. Given an iteration level curve l_n^i , the mortgage rate function $\bar{m}_n^i(x)$ is defined in a similar fashion as $\bar{m}_n^0(x)$. Iteration level curves l_n^i themselves are defined iteratively as solutions to

$$m_n = \mathcal{A}[\bar{m}_n^{i-1}](m_n, x), \quad i = 1, 2, \dots$$

In this case we have to run iterations but the complexity of the procedure still stays independent of the dimension of the state process (without regard to computation of expectations in \mathcal{A}), which makes the procedure more promising than the iteration procedures based on the computation of the mortgage rates given fixed interest rate values.

7 Note on Monte-Carlo Implementation

Although the complexity of the numerical procedure by itself, as we’ve seen, does not grow with the dimension of the state process, the problem of dimensionality might appear in the valuation of expectations itself. PDE approach, illustrated in Section 3.1, can be easily implemented in a one-factor setting. It is feasible to implement the same framework in the case two- or three-dimensional state processes. However, for higher dimensions the problem of the “curse of dimensionality” might make the problem computationally too hard.

To overcome the curse of dimensionality, one might want to use Monte Carlo simulation. Additionally, Monte Carlo method has a very important advantage for practitioners in the sense that one could very easily implement different kinds of interest rate models (e.g., jump-diffusion). However a “straightforward” Monte Carlo implementation implies computation of expectations at one given point (initial position of the state process), while for our iterative procedure we need to know the “whole” conditional expectation to find an appropriate level curve.¹²

The answer to this problem can be given by regression techniques which were used for Monte Carlo estimation of the optimal stopping time for American options pricing in [12], [11], [8]. The idea is based on the fact that conditional expectations from the L^2 space can be

¹²Or at least, we should know the conditional expectation in some domain which would apriori contain this level curve

represented by a linear combination of functions from a *countable basis* in this space (since L^2 is a Hilbert space). In practice, the expectations are approximated by a finite number of these basis functions.

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