On the Mortgage Rates implied by the Option-Based and Empirical Approaches

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November 15, 2003

Abstract

This paper presents a mortgage rate equation formulated as a fixed point equation and proposes a numerical scheme for it. The scheme is tested on the option-based and mortgage-rate-based approaches to mortgage modeling. This is the first attempt to consistently compare two approaches. The result shows a striking difference between them.

1 Introduction

A very important practical problem of a financial institution issuing mortgages is finding the "fair" mortgage rate. Mortgages offered with a lower rate may inflict heavy losses since the mortgage pool will be underpriced. If offering a higher mortgage rate, the institution may not be competitive (i.e., it may have no pool at all). If borrowers refinance using mainly information on available for refinancing mortgage rates (the web is abundant with "calculators" which tell you how much one saves if he/she refinances) then the current mortgage rate, clearly, depends on the whole future behavior of the mortgage rate. In this case the problem is to identify the mortgage rate process, i.e, to find a mortgage rate as a function of factors being used in the underlying model. Such an endogenous mortgage rate would be useful for an accurate prediction of prepayment rates. Pricing all mortgage securities is heavily dependent on this prediction.

The problem of modeling the mortgage rate is addressed exclusively empirically in the literature. For example, a popular choice is to assume that the mortgage rate is the 10-year Treasury yield plus some (exogenously specified) constant. Only Kau in his option-based model mentioned so called balancing condition to get the value of the endogenous (that is implied by the model) mortgage rate. No research was conducted to study endogenous mortgage rate process. This partially can be attributed to the fact that mortgage researchers did not apply tools to properly formulate a mortgage rate equation. The first general model of a mortgage with sub-optimal prepayment was developed by the author [6]. Based on this model we define in this paper the endogenous mortgage rate as a solution to a fixed-point problem. The author hopes that this paper inspire an interest of researchers in finance and mathematics to this complicated and practically important problem.

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2 A Mortgage

We consider the following contract. A borrower takes a loan of P_0 dollars at some initial time and assumes the obligation to pay scheduled coupons at rate $c_t \geq 0$ continuously for duration T of the contract. The loan is secured by the collateral of some specified real estate property, which obliges the borrower to make the payments. For a mortgage originated at time s, interest on the principal is compounded according to a contract rate m^t, where time dependence as the superscript t shows the time when this mortgage rate schedule was contracted. The superscript notation is chosen to stress that the mortgage rate is fixed for a life of the mortgage. If a mortgage is originated at time $t = 0$ we suppress the superscript 0 and use merely the notation m for the mortgage rate.

The borrower has the right to settle his/her obligation during an interval specified by contract[1] and prepay the outstanding principal $P(t)$ in a lump sum. If the borrower prepays then he is forced to pay concomitant transaction costs. The transaction cost process F_t is assumed to be a defined part of the mortgage model.

The coupon payment rate c_t is a deterministic function of time. We assume that the mortgage is fully amortized, i.e., $P(T) = 0$. In the case of the most popular level-payment, fixed rate mortgage, c_t is a constant. Other examples of fixed-rate mortgages with non-constant c_t are graduated payment mortgage and growing equity mortgage.

As can be seen, the outstanding principal $P(t)$ depends on the choice of the contracted mortgage rate m and the contracted coupon payments c_t. This dependence is suppressed for notational transparency.

3 Mathematical Apparatus

We formalize our setup by introducing a completed filtered probability space $(\mathbf{\Omega}, \mathbf{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbf{Q})$,[2] where the σ-algebra $\mathcal{G}_t$ represents all observations available to an investor at time t, $\mathbf{\Omega}$ is a set of all possible outcomes and $\mathbf{Q}$ is the probability on $\mathbf{G}$ ($\supseteq \bigcup_{t\geq 0} \mathcal{G}_t$). The prepayment time, for which we use notation τ, is then a positive stopping time on this filtered probability space (i.e., at any arbitrary time t we can tell if prepayment "occurred" given information $\mathcal{G}_t$).

We introduce information concerning only the timing of the prepayment as the filtration $\mathcal{D}_t = \sigma(\mathbb{I}_{\{\tau \leq u\}} | u \leq t)$. Now given $\mathcal{D}_t$ and the original filtration $\mathcal{G}_t$, we are interested in decomposing $\mathcal{G}_t$ into $\mathcal{D}_t$ and an additional filtration $\mathcal{F}_t$. Formally, it can be defined as a solution of the equality $\mathcal{G}_t = \mathcal{D}_t \vee \mathcal{F}_t$. We accept this complementary filtration $\{\mathcal{F}_t\}_{t\geq 0}$ as given. In the financial interpretation, $\{\mathcal{F}_t\}_{t\geq 0}$ is assumed to model the flow of observations available to the lender *prior* to the prepayment time τ. Given only this information $\{\mathcal{F}_t\}_{t\geq 0}$, the lender cannot anticipate the prepayment since he/she does not have complete information about the borrower (such as intention to move, to divorce, etc.) and/or the borrower does not prepay as soon as it is profitable to do so (see, e.g, Hayre and Rajan [8]). Thus, we assume that prepayment time τ is not an $\mathcal{F}_t$-stopping time. Throughout this paper we assume that all given processes are positive, $\mathcal{F}_t$-adapted and right-continuous.

The Fundamental Theorem of Arbitrage Pricing (see Harrison and Pliska [7]) states that absence of arbitrage implies that all securities are priced in terms of this short-rate process r_t and an

[1] Commercial mortgages, for example, often have a prepayment lockout period.

[2] We make throughout the usual technical assumption that this and all filtrations we work with later satisfy the "usual hypothesis" and are complete.

equivalent martingale measure. Therefore, it is convenient to assume that the probability $\mathbf{Q}$ in the introduced probability space is this martingale measure and, thus, all expectations in the paper will be taken under this martingale measure $\mathbf{Q}$ without reminder. The connection between real-world and martingale probability measures will be briefly discussed at the end of this section.

The following definition is standard (see Jeanblanc and Rutkowski [10]) and is well defined because $Q(\tau \le |\mathcal{F}_t) < 1$.

Definition 3.1 *The process* $\Gamma_t = -\ln(1 - Q(\tau \le t|\mathcal{F}_t))$ *is called the hazard process of the random time* τ. *Equivalently* $Q(\tau > t|\mathcal{F}_t) = e^{-\Gamma_t}$.

We assume that Γ_t is an increasing process. Because there is no particular convenient or special day for prepayment,[3] Γ_t is assumed to be continuous. Indeed, we make the slightly more restrictive assumption that the process Γ_t is an absolutely continuous process, i.e., $\Gamma_t = \int_0^t \gamma_\theta d\theta$ for some process γ_t, called the intensity of the random time τ. In this case the definition has certain similarities with the definition of the intensity of a Poisson process, thereby giving us intuition behind the name "intensity" of γ_t.[4]

The following theorem (see Goncharov [6]) gives us a formula for the mortgage price.

Theorem 3.1 *The mortgage price satisfy the following equation for* $t < \tau$:

$$M_t = P(t) + \mathbf{E}\left[\int\limits_t^T (m_s - r_s)P(s)e^{-\int\limits_t^s (r_\theta+\gamma_\theta)d\theta} ds \;\middle|\; \mathcal{F}_t\right]. \tag{1}$$

The last theorem shows that a mortgage can be viewed as a defaultable interest-rate swap on notional amount $P(t)$. The parties exchange interest payments calculated according to the scheduled (usually fixed) rate m_t and interest rate r_t up to possible "default" (i.e., prepayment in our case), the timing of which is driven be the intensity γ_t.

Later we will need to work with the borrower's liability L_t. We note that the distinction between the liability to the mortgagor and the price of asset to the investor is that at time of prepayment a holder of the security receives the outstanding principal $P(t)$, but the mortgagor pays $P(t)$ plus a transaction cost by F_t. Therefore the liability L_t can be defined as follows (see Goncharov [6] for details):

$$L_t = M_t + \mathbf{E}\left[\int\limits_t^T F_s\gamma_s e^{-\int\limits_t^s [r_\theta+\gamma_\theta]d\theta} ds \;\middle|\; \mathcal{F}_t\right] \tag{2}$$

Note on the real world and martingale measures. In the literature on mortgage valuation, authors often capture the stochastic nature of prepayments with an empirically estimated prepayment function of some state process such as the borrower's prepayment incentive, loan-to-value ratio, interest rates, etc. The common feature of this literature is that the prepayment function is assumed to be the same under the real world and martingale measures. Jarrow, Lando and Yu,

[3] For example, for defaultable mortgages with discrete payments it could be reasonable to assume that default times are distributed at due dates, in which case the due dates would be "special" for default. We work with continuous coupon cash flows.

[4] If τ is the first jump of a Poisson process with an intensity function $\gamma(t)$ then $Q(\tau > t) = e^{-\int_0^t \gamma(\theta)d\theta}$. Etymology of the term "intensity" (or "hazard rate") for $\gamma(t)$ came from the fact that for small $\triangle t$ we have $Q(t < \tau \le t + \triangle t|\tau > t) = \gamma(t)\triangle t + o(\triangle t)$ for almost all $t \in R_+$.

in their work [9] on default risk, argue that this equivalence is an example of an implicitly applied assumption of conditional diversification. Briefly (and rephrasing the authors so our wording is in terms of mortgage prepayment rather then default), the notion of conditional diversifibility requires that conditioning on the evolution of the state processes, the prepayment processes of borrowers are independent of each other. This captures the idea that once the systematic parts of prepayment risk have been isolated, the residual parts represent idiosyncratic, or borrower-specific, shocks that are uncorrelated across borrowers. Examples of such shocks include divorce, acquisition of a new job or its loss, advent of a new family member, etc. As the nonsystematic risk is not priced, it justifies the practice of using an empirically estimated prepayment function for valuation purposes.

For ourselves we can add that the creation of mortgage-backed securities, namely, the practice of pooling individual mortgages, can be seen as the "physical" implementation of the above argument. As such, we view the prepayment intensity γ_t as a model for prepayment. In general, we state "mortgage model" = "intensity model" = "prepayment model." Of course, question about how close the prepayment intensity is under the risk-neutral and physical probability measures deserves deep empirical research. ■

4 Mortgage Rate

It is of a vital practical interest for Wall Street to define the mortgage rate endogenously, i.e., to find what mortgage rate is implied by the current (riskless) yield curve and the prepayment behavior of a representative mortgagor. Consider the case of the fixed-rate mortgages, i.e., m^t is a mortgage rate of a fixed-rate mortgage originated at time t. Then this can be done through the postulate that at origination the value of a mortgage should be equal to the initial principal. For example, from (1), using the arbitrage argument that $M_0 = P_0$, we can get that the mortgage rate m^t of a fixed-rate default-free mortgage at time t is a weighted average of the risk-free interest rate over the life-time T of the contract, that is,[5]

$$m^t = \frac{\mathbf{E}\left[\int\limits_t^{T+t} r_s P(s-t) e^{-\int\limits_0^s (r_\theta + \gamma_\theta) d\theta} ds \,\middle|\, \mathcal{F}_t \right]}{\mathbf{E}\left[\int\limits_t^{T+t} P(s-t) e^{-\int\limits_0^s (r_\theta + \gamma_\theta) d\theta} ds \,\middle|\, \mathcal{F}_t \right]}. \tag{3}$$

Unfortunately, this is an equation rather than a formula because at least $P(s)$ and thus the right-hand side of (3) depend on the mortgage rate too. The prepayment intensity γ_t may also depend on the mortgage rate. This dependence can be quite different for different approaches. The prepayment intensity γ_t can be modeled in such a way that it depends *only* on the contract mortgage rate m^t for all $s \in [t, T+t]$ (this is the case of the traditional option-based approach where γ_t depends on m^t implicitly through the mortgage price or the borrower's liability) or it can depend on the future mortgage rates m^s as well (the MRB approaches). The latter case (MRB) greatly complicates the problem because equation (3) is a functional equation rather than just a nonlinear equation in one variable as in the former case (the option-based approaches).

[5] Mortgages are mostly traded in pools. Therefore, if one assumes heterogeneity of borrowers, the denominator and the numerator of equation (3) should be "averaged" over a possible "risk-neutral" distribution of borrowers' prepayment processes γ_t.

If γ_t and r_t depend on diffusion processes and, additionally. γ_t depends on the price of the mortgage M_t (or the borrower's liability L_t; this is a framework for the option-based approaches.) then we have the following theorem on the existence of a solution.

Theorem 4.1 *A continuous solution $m(x)$ of the mortgage rate equation exists with specification:*[6]

$$m(x) = \frac{\mathbf{E}\left[\int_0^T r_s P(s; m(x)) e^{-\int_0^s [r_\theta + \gamma_\theta(M_t^{m(x)})] d\theta} ds \,\middle|\, X_0 = x\right]}{\mathbf{E}\left[\int_0^T P(s; m(x)) e^{-\int_0^s [r_\theta + \gamma_\theta(M_t^{m(x)})] d\theta} ds \,\middle|\, X_0 = x\right]} \tag{4}$$

5 Numerical Scheme for the Mortgage Rate Equation

In this section we assume that the only stochastic process in the model is the instantaneous interest rate r_t. We consider a time-homogeneous model. Thus the mortgage rate process m^t is merely a function of the interest rate r_t, i.e., $m^t = m(r_t)$. For the right hand side of the mortgage rate equation (3) we use notation $\mathcal{A}(m)$, where $m = \{m(r)|r \geq 0\}$ is the mortgage rate function. The mortgage rate equation itself can be written as $m = \mathcal{A}(m)$ in this case, i.e., a solution to the mortgage rate equation is a fixed point of the operator $\mathcal{A}$. This equation can be written as a family of uncoupled scalar equations $m(r) = \mathcal{A}(m(r)),\ r \geq 0$, for the traditional or reduced option-based approaches, but it is essentially a functional equation for, say, (non-empirical) MRB approaches. The numerical scheme below is formulated for the latter case, but it is equally applicable for the option-based approaches if one desires to find the mortgage rate as a function of the interest rate.

The refinancing incentive at time t is assumed to be a function of the contract and current mortgage rates, i.e., $\Pi_t = \Pi(m_0, m_t)$. Since it is not likely that a borrower will refinance to a mortgage with the higher mortgage rate we postulate that if $m_0 \leq m_1$ then $\Pi(m_0, m_1) = 0$. We find it natural to assume monotonicity of the mortgage rate, i.e., $m(r_1) \leq m(r_2)$ if $r_1 \leq r_2$. A model that violates this assumption would look "suspicious" since the fall in the interest rates would imply an increase in mortgage rates having all factors the same. Next, we assume that $\Pi(m_0, m_2) \leq \Pi(m_0, m_1)$ for $m_1 \leq m_2$, i.e., the lower the available-for-refinancing mortgage rate the more likely borrowers refinance their mortgages (recall that the prepayment intensity is an increasing function of the refinancing incentive Π_t). Combining these monotonicity assumptions on $m(\cdot)$ and $\Pi(m_0, \cdot)$ we get that the prepayment intensity of some particular borrower is a decreasing function of the interest rate r_t, just as one would intuitively expect. Finally, we assume that the prepayment intensity is a constant[7] γ_o for refinancing incentive Π_t less than the transaction cost.

From the above assumptions it follows that if one wants to know, say, the value of some mortgage-backed security for a specific interest rate r^*, then he/she needs to know only the mortgage rates for the values of the interest rates not greater than r^*, i.e., the set $\{m(r)|r \leq r^*\}$. In

[6] We use notation $M_t^{m(x)}$ and $P(t; m(x))$ for the mortgage price to remind the reader about the ulterior dependence of the mortgage price on the mortgage rate m.

[7] γ_o can be a function of the interest rate r_t and/or time since the origination of the mortgage to model a burnout effect. It does not change the method below.

particular, $m(r^*)$ can be found from the equation (3) knowing only the set $\{m(r)|r < r^*\}$. This idea is behind the following numerical scheme.

Let $\mathcal{M} = \{r_i|\ i = 0, 1, 2, ..., N\}$ be a grid over the interest rate values and let $\triangle_i = r_i - r_{i-1}$ be its step sizes. Our purpose is to find approximations of $m(r_i)$, $i = 0, 1, 2, ..., N$ for which we use notation m_i. We do not specify the numerical method for finding the expectations in the mortgage rate equation (3). If this method requires the knowledge of $m(r)$ for intermediate r's in the interest rate grid, then we assume that they are defined with the help of some interpolation based on $\mathcal{M}$. Assume that r_0 is the lowest value of the interest rate (i.e., zero for most models). Therefore m_0 can be found as a solution of (3) with the constant γ_0 instead of the process γ_t, i.e., in this case we expect exogenous prepayment only (e.g., sell of the house due to divorce or new job in another city). This solution can be found with the help of an iterative procedure. Let m_0^0 be our initial guess, then the sequence m_0^n, $n = 1, 2, ...$ is determined inductively:

$$m_0^{i+1} = \mathcal{A}(m_0^i) = \frac{\mathbf{E}\left[\int_0^T r_s P(s; m_0^i) e^{-\int_t^s r_\theta d\theta - \gamma_o s} ds \;\middle|\; r_0\right]}{\mathbf{E}\left[\int_0^T P(s; m_0^i) e^{-\int_t^s r_\theta d\theta - \gamma_o s} ds \;\middle|\; r_0\right]}, \tag{5}$$

where m_0^i enters the right hand side through the definition of $P(t)$ only. Now we expect that $m_0^i \longrightarrow m_0$ for $i \to \infty$. For this (and for all that follow) iteration we can employ various accelerations (such as Aitkins) to get better performance.

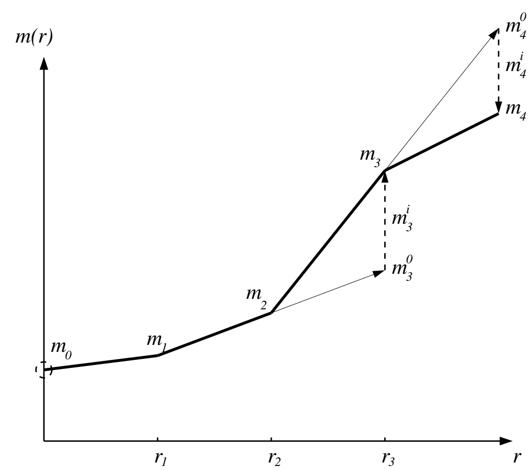

Figure 1: Consecutive application of the iterative scheme (5).

From our specification we can conclude that to find m_n for $n = 1, 2, ..., N$ we need to know only values of m_k, $k = 0, ..., n-1$. Indeed, for r_k, $k = n, ..., N$ (i.e., $r_k \leq r_n$) we have $m(r_k) \geq m(r_n)$ and, thus, γ_t is just a constant γ_o for these values of the interest rate. Therefore we can find m_n for $n = 1, 2, ..., N$ in order. If we know the first n values of m_n then the next mortgage rate m_{n+1} can be found with the help of the iterative procedure

$$m_{n+1}^{i+1} = \mathcal{A}(m_{n+1}^i; \{m_k|\ k = 0, ..., n\})$$

$$:= \frac{\mathbf{E}\left[\int_0^T r_s P(s; m_{n+1}^i) e^{-\int_t^s [r_\theta + \gamma(m_{n+1}^i; \{m_k | \; k=0,...,n\})] d\theta} ds \; \middle| \; r_{n+1}\right]}{\mathbf{E}\left[\int_0^T P(s; m_{n+1}^i) e^{-\int_t^s [r_\theta + \gamma(m_{n+1}^i; \{m_k | \; k=0,...,n\})] d\theta} ds \; \middle| \; r_{n+1}\right]}.$$

Finding m_n's successively we provide ourselves with a good initial guess for this iterative procedure. From econometric considerations we expect the function $m(r)$ to be continuous for a reasonable model. Therefore m_n should not be "far" from m_{n-1}. If the underlying model gives differentiability of $m(r)$ then we can use Euler's rule to take the even better guess $m_{n+1}^0 = m_n + (m_n - m_{n-1})(\triangle_{n+1}/\triangle_n)$. This guess reduces the number of iterations needed for a given precision.

Note on the interest rate grid. Suppose the expectations in the mortgage rate equation (3) are evaluated on the grid $\mathcal{R} = \{r_i | \; i = 0, 1, 2, ..., M\}$ with the characteristic step size $\triangle_{\mathcal{R}}$. Assume that the mortgage rate function $m(r)$ is twice differentiable and is approximated on the grid $\mathcal{M} = \{r_i | \; i = 0, 1, 2, ..., N\}$ with the characteristic step size $\triangle_{\mathcal{M}}$. If the method employed to find the expectations is of the second order then it is reasonable to take $\mathcal{M} = \mathcal{R}$. However, if the method is of the first order then we can calculate the mortgage rate function $m(r)$ on the grid $\mathcal{M} \subset \mathcal{R}$ with the characteristic step size $\triangle_{\mathcal{M}} = O(\sqrt{\triangle_{\mathcal{R}}})$ interpolating the values of $m(r)$ to $\mathcal{R}$ (for calculation of the expectations) with the help of a higher order spline. ■

6 Specification of Refinancing Incentive

As we can see from (1), γ_t determines the price of a mortgage. It governs the timing of prepayment, the most specific characteristic of mortgage modeling. Therefore its specification is a cornerstone of mortgage pricing. Let Π_t be a scalar quantity which measures the refinancing incentive of the borrower. We assume that refinancing becomes profitable as soon as the refinancing incentive is higher then the refinancing transaction cost, i.e., $\Pi_t > F_t$. The larger the refinancing incentive the more likely the borrower will refinance his/her mortgage. Therefore, to emphasize this dependence, we single out variable Π in the notation of the prepayment intensity, i.e., the process $\gamma_t = \gamma_t(\Pi)$ is a function of Π for fixed $(t, \omega) \in [0, T] \times D$. The function $\gamma_t(\cdot)$ measures how "fast" the borrower is expected to use a refinancing opportunity. Existing mortgage models mostly assume that $\gamma_t(\Pi)$ is a constant as a function of Π up to a point where Π is less then transaction costs and a greater constant or growing linear function after that. The actual form of γ_t dependence on Π is borrower specific and, therefore, it is a (important!) matter of statistical research. Once we find $\gamma_t(\cdot)$, it should be stable with respect to changes in economical variables and structural changes in market. These changes are reflected in the refinancing incentive Π itself. According the way this quantity is constructed we can classify the models in the literature for valuation of mortgage contracts into one of two categories: the *option-based* models and the empirical *mortgage-rate-based* models (empirical MRB for short).

6.1 MRB Approaches

A simple way to measure the prepayment incentive is to compare the contract mortgage m^0 with an available for refinancing mortgage rate m^t. On the web, for example, one can find a lot of calculators which can say if it is profitable to refinance (and how much one saves) on the base of mortgage rates information. The approach is intuitively clear and assumes that the borrower considers refinancing to another mortgage of the same type. As Hayre and Rajan [8] report, the fixed-rate 30-year mortgage has retained its popularity as the refinancing vehicle of choice for mortgagors with an existing 30-year loan. That gives a rational for the MRB definition of the refinancing incentive.

The refinancing incentive Π_t can be the difference $m^0 - m^t$ (models by Kariya and Kobayashi [11], Kariya, Pliska and Ushiyama [12] and Schwartz and Torous [14, 15] assume this specification), the rational expression m^0/m^t (as was argued in Hayre and Rajan [8] and Richard and Roll [13], it is able to capture refinancing incentive better then the difference), or it can be given by direct computation of how much the borrower will save on coupon payments with refinancing (Deng [3], Deng and Quigley [4], Deng, Quigley and Van Order [5]). Depending on the type (e.g., a 30-year or 15-year mortgage rate) of the mortgage rates m^t and m^0 considered, the expressions above approximate/evaluate potential savings from refinancing a m^0-fixed-rate mortgage with a (30-year or 15-year mortgage rate) m^t-fixed-rate mortgage.

The mortgage rate can be defined exogenously (empirically). In this (empirical MRB) case a common choice is to model the mortgage rate as the 10-year Treasury note yield plus a constant. As reported in Boudoukh, Richardson, Stanton and Whitelaw [2], the 10-year Treasury note yield has correlation of 0.98 with the mortgage rate. Belbase and Szakallas [1] found that Libor swap rates are better predictors than the Treasury yield (the correlation of 30-year mortgage rate vs. 10-year Libor swap rate and vs. 10-year Treasury yield are .983 and .915 respectively in their study).

We propose to extend the MRB approach by defining the mortgage rate process endogenously. After solving a mortgage rate equation we have the endogenous prepayment intensity γ_t (i.e., prepayment model). As a result we get a model which is "immuned" to structural changes in the underlying variables.

MRB approaches allow easy fit to prepayment data as opposed to option-based approaches. The mortgage rate data are readily available, thus the refinancing incentive is available without additional calculations. We can use this data together with prepayment data to calibrate the prepayment intensity function.

6.2 "Traditional" Option-Based Approach

The MRB approach is "naive:" the MRB refinancing incentive $\Pi_t(m^0, m^t)$ does not depend on, for example, volatility of the interest rates.[8] As is well known from option pricing theory (mortgages have embedded options!), the value of an option is sensitive to changes in volatility of an underlying asset. We may expect that Π_t decreases if the volatility increases (keeping the other factors the same) since the probability of the option being (deeper) "in-the-money" is higher. This feature is fully incorporated in the following approach, which is based on the option pricing idea.

Intuitively saying the price of, say, an American option can be found under an assumption that a holder of the option exercises it as soon as doing so is profitable, i.e., the payoff is greater then

[8] One can include this dependence (and others), but it would be an ad-hoc approach.

the price of the option. In the context of mortgage pricing, if a borrower prepays the mortgage, he/she must pay the outstanding principal $P(t)$ plus (perhaps substantial) transaction costs and, in return, he/she is liberated from his/her obligations to pay coupons, i.e., he/she "gets back" his/her liability L_t. This idea stands behind the option-based definition of the refinancing incentive, where refinancing incentive $\Pi_t = L_t - P(t)$. The traditional option-based approach implies that $\gamma_t = \gamma_t(L_t - P(t))$ and, thus, (2) is not a formula but an equation. (see Goncharov [6] on existence and uniqueness of a solution).

Note on the liability specification. The liability L_t is not just the discounted value of the cash flow c_t; it takes into account future prepayment opportunities. It is important to note the presence of γ_t and F_t in the definition of the liability, i.e., the borrower is aware of the possibilities to miss prepayment opportunities in the future and takes into account the future transaction costs associated with the prepayment. This feature is sometimes overseen in the option-based mortgage modeling literature. ■

After we solve equation (2), the prepayment intensity γ_t is an $\mathcal{F}_t$-adapted processes, we are in the framework of section 3, and can find the price of the mortgage or related mortgage security.

The traditional option-based approach does not require one to model the mortgage rate process. The knowledge of the contract rate c_t (i.e., the mortgage rate m^t at origination of the mortgage) is sufficient. However, things get more complicated when we calibrate the prepayment intensity function. The borrower liability is not available to us without additional calculations and, therefore, the refinancing incentive data are not "free" as they were in the case of the MRB approaches. Therefore the prepayment intensity calibration procedure should be done together with the liability valuation, and this is numerically an expensive procedure.

6.3 MRB vs. Option-Based Specification of Refinancing Incentive

An option-based measure of refinancing incentive is based on a comparison of outstanding principal $P(t)$ and the liability of the mortgagor. Whatever kind of a loan the mortgagor chooses to pay off, the price of the loan at the origination should coincide with $P(t)$. On the other hand, we have to keep in mind that the borrower probably refinances to another mortgage of the same type and, thus, prolongs the time of repayment of the loan (that may be undesirable for the mortgagor). However, from the option-based approach point of view, the borrower is indifferent to the length of repayment time.

Example. The following simple example emphasizes this point. Let us consider a 30-year mortgage. Assume that the risk-free interest rate is deterministic and equals the constant r_1 from 0 to 15 years, then 0 from 15 to 30 years and the constant r_2 from 30 years on. The prepayment intensity γ_t is arbitrary, but bounded.

From the mortgage rate equation (3) we can conclude that the mortgage rate at time 0 is less then r_1, i.e., $m^0 < r_1$ (the particular value of m^0 depends on the prepayment intensity γ_t which we do not specify here), and $m^0 \to +\infty$ monotonically as $r_1 \to +\infty$. From the same equation we can see that at time $t = 15$ the mortgage rate m^{15} is monotonically increasing to $+\infty$ as $r_2 \to +\infty$. The liability L_{15} is higher then $P(15)$ because the interest rate r_t is zero for $t \in (15, 30)$. Moreover, $L_t - P(t)$ can be made larger then any specified transaction cost by choosing r_1 large enough. This means that at time $t = 15$ it is profitable to refinance the mortgage from the option-based point of view for that value of r_1. However, choosing r_2 large enough we can have $m^{15} > m^0$. Therefore, at time $t = 15$ refinancing to another 30-year mortgage is unprofitable from MRB point of view. It is unlikely that the borrower will follow "option-based" advise which causes him to refinance to another mortgage with a higher rate. ■

This shows that the option-based measure of refinancing incentive may not be adequate. This measure neglects the borrower's preferences and ability to pay a certain cash flow. The outstanding principal $P(t)$ is compared with the liability, which is based on the behavior of economic variables over the period of time $[t, T]$, thus making the traditional option-based refinancing incentive measure "comparable"[9] with refinancing the current mortgage into a $(T-t)$-year fixed-rate mortgage (which is valid over the same period of time), the choice of which is not readily available to a residential mortgagor.

Another reason why the MRB choice of incentive measure may be more appropriate then the option-based measure is that the typical residential borrower is not financially astute, as sophisticated financial models are not freely accessible for the general public. A mortgagor calculates his/her amount of savings (true and ultimate measure of incentive to prepay his/her mortgage!) with refinancing, using currently available alternative mortgage rates and comparing monthly payments implied by these rates (like in Richard and Roll [13], from which the authors get m^0/m^t as an appropriate incentive measure). Such "refinancing" calculators are very popular, easily accessible on the world wide web and are often used as an advertisement for refinancing. This may introduce a bias in prepayment behavior of mortgagors which may work in favor of the MRB approach.

Note on close-to-maturity mortgages. The option-based approach can be well-suited for close-to-maturity mortgages since borrowers can use low short interest rate situation, (the mortgage rates can still be high) to repay their mortgages using their own savings or refinancing to balloon or adjustable-rate mortgages.

7 Application

The numerical scheme was tested on the interest rate and $\gamma(\cdot)$ specifications used by Stanton in [16]. The coefficients of the CIR interest rate model

$$dr_t = a(b - r_t)dt + \sigma\sqrt{r}dW_t$$

are taken as follows:[10]

$$\begin{aligned} a &= 0.17203 \\ b &= 0.13546 \\ \sigma &= 0.11425. \end{aligned}$$

As we can see the long run mean is 13.5% under the martingale measure. Taking into account the market price, which is implied to be $\lambda = -1.06477\sqrt{r}$ in [16], the long run mean is 7.9% under the

[9] Actually, it is more precise to say that the traditional option-based approach assumes that a mortgage is refinanced by taking a loan from a bank at a risk-free rate. This is not realistic and contributes to the overestimated transaction costs in option-based models.

[10] The interest rate model in Stanton [16] was stated under the physical probability measure, therefore the CIR parameters there are different from ours. Our coefficients and the market price of risk deducted from this parameters and the pricing PDE.

physical measure. We remind the reader that the prepayment intensity in [16] was assumed to be a step-function

$$\gamma(\Pi_t) = \begin{cases} \lambda & , \quad \text{if } \Pi_t \leq P(t)F \\ \rho + \lambda & , \quad \text{otherwise,} \end{cases}$$

where λ is an exogenous part of the prepayment intensity, i.e., it refers to prepayments due to exogenous reasons (such as new job, divorce, etc.) and ρ is an endogenous part of the prepayment intensity, i.e., it refers to prepayments due to financially based decisions.

We choose the transaction cost to be 30% of the outstanding principal, i.e., F in the above specification of γ is 0.3.[11]

We consider the traditional option-based approach, i.e., $\Pi_t = L_t - P(t)$ (exactly as in Stanton [16]) and the endogenous MRB with $\Pi_t = P(t)c(m^0)/c(m^t)$,[12] where $c(m)$ is a coupon rate as determined by the contracted mortgage rate m. The results "the option-based vs. MRB approaches" are given on Figure 2. We can see that the "S"-shaped graphs of the MRB mortgage rates are quite different from the graphs of the option-based mortgage rates.

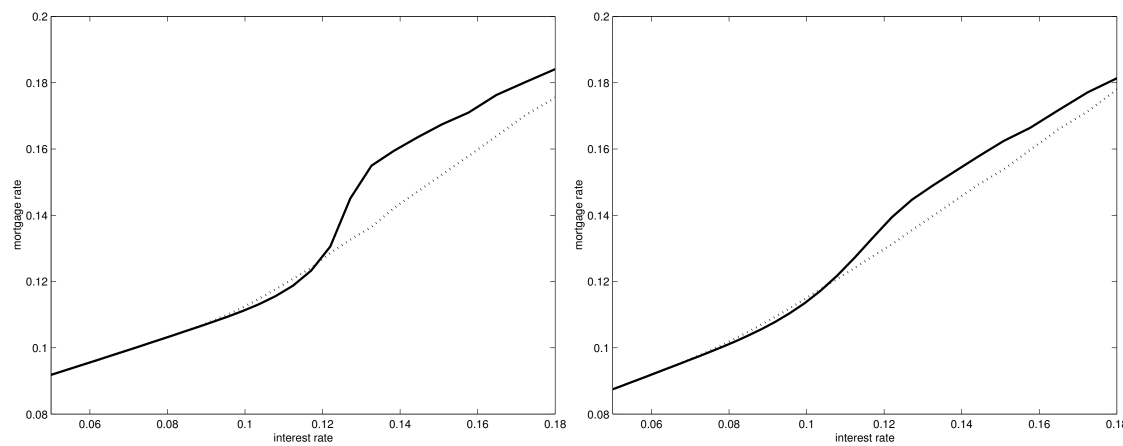

Figure 2: The mortgage rates as implied by the MRB (solid line) and the option-based (dotted line) approaches. The graphs are given for two different values of λ: 0 (left graph) and 0.05 (right graph); $\rho = 2$.

The increase of the exogenous part of the prepayment intensity λ smooths the graphs of both approaches, but the change is more pronounced and is more complex in the MRB graph. This can be seen on Figure 3.

The difference is even more striking in sensitivity to variation of the endogenous part of the prepayment intensity ρ. This is illustrated on Figure 4.

[11] Stanton estimated F to be concentrated in the range of $30 - 50\%$ with a mean value of 41%.

[12] The fraction $c(m^0)/c(m^t)$ shows the proportion of how much is saved with refinancing, i.e., it is a relative quantity. Multiplying by $P(t)$ we express the savings in terms of an absolute quantity.

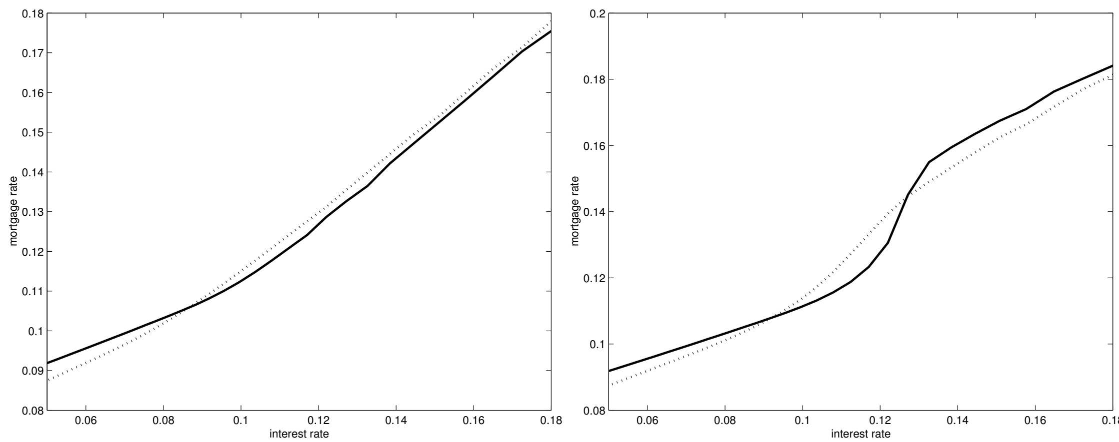

Figure 3: The mortgage rates as implied by the option-based (left graph) and the MRB (right graph) approaches. The graphs are given for two values of λ: 0 (solid lines) and 0.05 (dotted lines); $\rho = 2$.

8 Mortgage Securities

The mortgages are not being traded individually. To reduce a risk of early exercise, mortgages are pooled and, then, securities distributing cashflow from the pool are issued. Such securities, called mortgage-backed securities (MBS), remove prepayment uncertainty to some extent but introduce a strong history dependence in the form of so-called burnout.[13]

One simple (not model-based) way to model the burnout effect is purely empirical. For example, Schwartz and Torous [14] assume that a pool consists of borrowers with homogeneous characteristics and therefore are forced to take burnout (recall that burnout is a product of a pool's *heterogeneity*) into account as the explanatory variable $\ln[P(t)/P^*(t)]$, where $P(t)$ represents the dollar amount of the pool outstanding at time t, while $P^*(t)$ is the pool's principal that would prevail at t in the absence of prepayments but reflecting the amortization of the underlying mortgages. The adequacy of this approach is questionable because the burnout effect is "non-Markovian"; it should depend on the whole history path of prepayments. The mortgage rate process can be handled exactly as in the case of one mortgage in this case.

The modelling approach (e.g., Stanton [16]) is based on the observation that one particular mortgage does not have burnout by definition. Therefore, using conditional independence of individual mortgages, one can incorporate burnout into a model by modelling the prepayment behavior for individual mortgages and constructing a pool's prepayment behavior as a combination of individual prepayments. Let a mortgage pool consist of N groups of distinct borrowers (N may coincide with the number of borrowers in the pool, although, clearly, it is not computationally practical). All borrowers in a particular group has the same prepayment intensity γ_t^i, where i is an

[13]If the pool experienced a wave of prepayment then it is likely that most of those who used this refinancing opportunity are "fast," financially astute borrowers, whereas most of those who remain are "slower" borrowers. Therefore, in the presence of another refinancing opportunity, the pool is expected to be less active — the pool is "burned out effect".

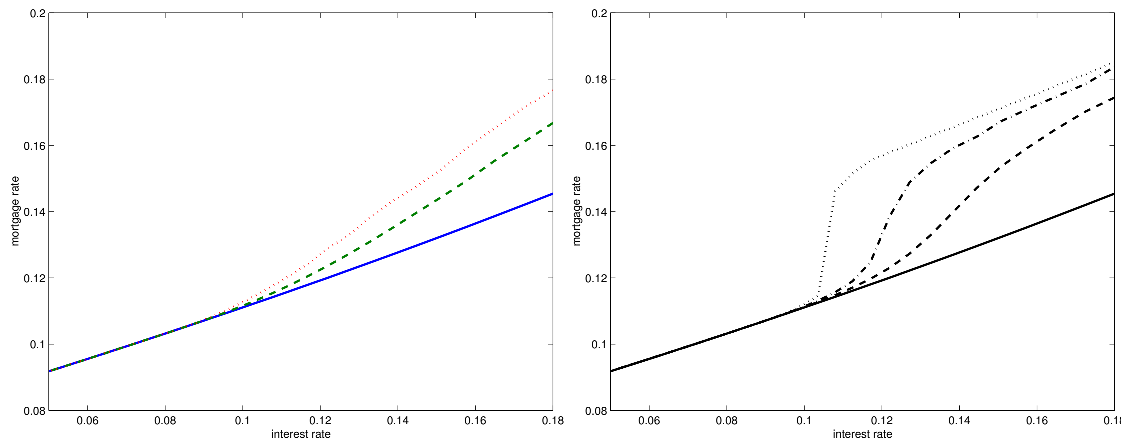

Figure 4: The mortgage rates as implied by the option-based (left graph) and the MRB (right graph) approaches. The graphs are given for four values of ρ: 0 (solid lines), 0.6 (dashed lines), 2 (dot-dashed lines), and ∞ (dotted lines); $\lambda = 0$. On the option-based graph the cases $\rho = 2$ and $\rho = \infty$ are practically indistinguishable.

index of the group ($i = 1, ..., N$). Let $P^i(t)$ be a total outstanding principal of the group i. Then the price of the mortgage pool is

$$M_t = \sum_{i=1}^{N} P_t^i + \sum_{i=1}^{N} \mathbf{E}\left[\int_t^T (m_s - r_s) P^i(s) e^{-\int_t^s (r_\theta + \gamma_\theta^i) d\theta} ds \,\middle|\, \mathcal{F}_t\right] \tag{6}$$

Using the arbitrage argument we can conclude that $M_0 = \sum_{i=1}^N P_0^i$ the mortgage rate is defined by the following equation

$$m^t = \frac{\sum_{i=1}^{N} \mathbf{E}\left[\int_t^{T+t} r_s P^i(s-t) e^{-\int_0^s (r_\theta + \gamma_\theta^i) d\theta} ds \,\middle|\, \mathcal{F}_t\right]}{\sum_{i=1}^{N} \mathbf{E}\left[\int_t^{T+t} P^i(s-t) e^{-\int_0^s (r_\theta + \gamma_\theta^i) d\theta} ds \,\middle|\, \mathcal{F}_t\right]}. \tag{7}$$

However, in reality borrowers in a pool are better represented not by "discrete groups" as above but by some "continuous" distribution. Therefore, calculations can be reduced significantly if we use some quadrature instead of summations in (6) and (7). Let $\{\gamma_t^\xi\}_{\xi \in [0,1]}$ is a (one-dimensional for simplicity) set of possible prepayment intesities. Assume that this set is represented in the pool with the density function $\omega(\xi)$. In particular, the above consideration implies $\omega_\xi = \sum_i \omega_i * \delta(\xi - \xi_i)$, where ω_i is a proportion (in dollars) of the group i in the pool and $\delta(x)$ is a usual delta-function.

Then we have the following analog of (7):

$$m^t = \frac{\int\limits_a^b \mathbf{E}\left[\int\limits_t^{T+t} r_s P(s-t) e^{-\int\limits_0^s (r_\theta + \gamma_\theta^\xi) d\theta} ds \;\middle|\; \mathcal{F}_t \right] \omega(\xi) d\xi}{\int\limits_a^b \mathbf{E}\left[\int\limits_t^{T+t} P(s-t) e^{-\int\limits_0^s (r_\theta + \gamma_\theta^\xi) d\theta} ds \;\middle|\; \mathcal{F}_t \right] \omega(\xi) d\xi}. \tag{8}$$

Using a quadrature to estimate integrals with respect to ξ it is a good idea to take into account structure of the function $\omega(\xi)$. Such information can help to identify an optimal non-uniform grid on which the integrand will be sample (i.e., the expectations in (8) will be evaluated). This technique, called Gaussian quadrature, can be found in many books on numerical integration (see, for example, Press, et al [17] where the reader can find a code for the problem).

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