Midterm 1

MAS 3105 Linear Algebra

Student's Name: Solution

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Do all problems on a separate piece of paper. Show your work. Each problem is worth 8 points.

1. Write down the augmented matrix corresponding to the system of linear equations and find the value(s) of h so that the system is consistent

$$\begin{aligned} x_1 \;+ x_2 + \; x_3 &= 2 \\ x_2 - \; x_3 &= 0 \\ 2x_1 + \; x_2 + \; 3x_3 &= h \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -1 & 0 \\ 2 & 1 & 3 & h \end{array}\right] \xrightarrow{-2R_1+R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & h-4 \end{array}\right] \xrightarrow{R_2+R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & h-4 \end{array}\right] \to h-4=0 \Rightarrow \boxed{h=4}$$

2. Let

$$A = \begin{bmatrix} -3 & 1 & 0 & -1 \\ 1 & 0 & 1 & -1 \end{bmatrix}.$$

(a) Find the solution set of the nonhomogeneous system $A\vec{x} = \vec{b}$ as parametrized vectors where

$$\vec{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

2.(a)

$$\left[\begin{array}{cccc|c} -3 & 1 & 0 & -1 & 0 \\ 1 & 0 & 1 & -1 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cccc|c} 1 & 0 & 1 & -1 & 1 \\ -3 & 1 & 0 & -1 & 0 \end{array}\right] \xrightarrow{3R_1+R_2 \to R_2} \left[\begin{array}{cccc|c} 1 & 0 & 1 & -1 & 1 \\ 0 & 1 & 3 & -4 & 3 \end{array}\right]$$

So, $x_1 = -x_3 + x_4 + 1$, $x_2 = -3x_3 + 4x_4 + 3$, $x_3 = x_3$, $x_4 = x_4$

$$\Rightarrow X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -x_3+x_4+1 \\ -3x_3+4x_4+3 \\ x_3 \\ x_4 \end{bmatrix}$$

So $X = \begin{bmatrix} 1 \\ 3 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ -3 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 4 \\ 0 \\ 1 \end{bmatrix}$

(b) Find the solution set of the homogeneous equation $A\vec{x} = \vec{0}$ as the span of a finite set of linearly independent vectors.

2.(b)

$$\begin{bmatrix} -3 & 1 & 0 & -1 \\ 1 & 0 & 1 & -1 \end{bmatrix} \xrightarrow[\text{calculations}]{\text{Same}} \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 3 & -4 \end{bmatrix}$$

So $x_1 = -x_3 + x_4$, $x_2 = -3x_3 + 4x_4$, $x_3 = x_3$, $x_4 = x_4$

So $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -x_3+x_4 \\ -3x_3+4x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -3 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 4 \\ 0 \\ 1 \end{bmatrix}$

So, the solution set $= \text{span}\left\{ \begin{bmatrix} -1 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 0 \\ 1 \end{bmatrix} \right\}$

3. Determine whether the set of vectors is linearly independent, and justify your answer.

(a)

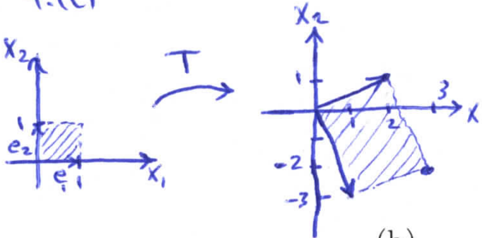
$$\left\{ \underbrace{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}_{v_1}, \underbrace{\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}}_{v_2}, \underbrace{\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}}_{v_3} \right\} \rightsquigarrow \text{linearly dependent}$$

Since,

$2v_1 - 2v_2 - v_3 = 0$

Extra points:

4.(c)



4.(d) $T\left(\begin{bmatrix}5\\-3\end{bmatrix}\right) = \begin{bmatrix}2 & 1\\1 & -3\end{bmatrix}\begin{bmatrix}5\\-3\end{bmatrix} = \begin{bmatrix}7\\14\end{bmatrix}$

$T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}2 & 1\\1 & -3\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}3\\-2\end{bmatrix}$

linearly independent, since if $\alpha u_1 + \beta u_2 = 0$

So $\begin{bmatrix}0\\-\alpha\\\alpha\\-\alpha\end{bmatrix} + \begin{bmatrix}-\beta\\\beta\\-\beta\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix} \Rightarrow \begin{bmatrix}-\beta\\-\alpha+\beta\\\alpha-\beta\\-\alpha\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$ $\to -\beta = 0 \Rightarrow \beta = 0$, $\to -\alpha = 0 \Rightarrow \alpha = 0$

So $\boxed{\alpha = \beta = 0}$

(b)

$$\left\{ \underbrace{\begin{bmatrix}0\\-1\\1\\-1\end{bmatrix}}_{u_1}, \underbrace{\begin{bmatrix}-1\\1\\-1\\0\end{bmatrix}}_{u_2} \right\}$$

4. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation satisfying

$$T(e_1) = 2e_1 + e_2, \quad \text{and} \quad T(e_2) = e_1 - 3e_2.$$

Answer questions (a),(b),(~~c~~) for this T.

extra credit (2 pts each)
(c) sketch the image of the square spanned by $e_1 + e_2$
(d) What is $T\left(\begin{bmatrix}5\\-3\end{bmatrix}\right)$?
$T\left(\begin{bmatrix}1\\1\end{bmatrix}\right)$ is also OK

(a) Find a matrix representation of T. $\begin{bmatrix}2 & 1\\1 & -3\end{bmatrix}$

(b) What is $T(5e_1 - 3e_2)$? $= T\left(\begin{bmatrix}5\\-3\end{bmatrix}\right) = \begin{bmatrix}2 & 1\\1 & -3\end{bmatrix}\begin{bmatrix}5\\-3\end{bmatrix} = \begin{bmatrix}7\\14\end{bmatrix}$

5. Give examples of each of the following.

(a) A nontrivial linear combination of the following vectors that equals zero, showing that the vectors are linear dependent.

$$-v_1 + v_2 + v_3 = 0 \qquad \left\{ \underbrace{\begin{bmatrix}-1\\0\\1\end{bmatrix}}_{v_1}, \underbrace{\begin{bmatrix}1\\1\\1\end{bmatrix}}_{v_2}, \underbrace{\begin{bmatrix}-2\\-1\\0\end{bmatrix}}_{v_3} \right\}.$$

(b) A vector that is not in the span of

$$\left\{ \begin{bmatrix}-1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix} \right\}.$$

$\begin{bmatrix}1\\0\\0\end{bmatrix}$, since if $\alpha\begin{bmatrix}-1\\0\\1\end{bmatrix} + \beta\begin{bmatrix}0\\1\\0\end{bmatrix} + \gamma\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$. So $\alpha = \beta = \gamma = 0$. Therefore, they are linearly independent and then $\begin{bmatrix}1\\0\\0\end{bmatrix} \notin \text{span}\left\{\begin{bmatrix}-1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$

6. Answer true or false. Justify your answers. (If the answer is "false", give a counter-example. If the answer is "true" briefly explain why it is true.)

(a) If the column vectors of a matrix A are linearly independent, then the corresponding linear transformation is one-to-one.

True, it implies that # pivots = # columns so there are no free variables and the corresponding linear transformation is one-to-one

(b) If the matrix transformation for A is one-to-one, then the number of pivots of the row-echelon form of A equals the number of rows of A.

False, $A = \begin{bmatrix}1 & 0\\0 & 1\\0 & 0\end{bmatrix} \to$ one-to-one since # columns = # pivots and no free variable. But # pivots $\neq$ # rows

(c) If $Ax = b$ has a solution, then so does $Ax = 2b$.

True, suppose that $\tilde{x}$ is a solution of $Ax = b$. So, $A\tilde{x} = b$. Now, $A(2\tilde{x}) = 2(A\tilde{x}) = 2b$. So $2\tilde{x}$ is a solution of $Ax = 2b$

(d) If A and B are two 2×2 matrices then $AB = BA$.

False, $A = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$, $B = \begin{bmatrix}1 & 0\\1 & 0\end{bmatrix}$

then, $AB = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$, $BA = \begin{bmatrix}1 & 0\\1 & 0\end{bmatrix}$ $\Rightarrow AB \neq BA$