Midterm II MAS 3105 Linear Algebra

Student's Name: Solution

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Do all problems on a separate piece of paper. Show your work. Each problem is worth 8 points.

(9 points) 1. Determine which of the following matrices are invertible, and if they are invertible find the determinant and the inverse.

(a)
$$A = \begin{bmatrix} -3 & 1 \\ 2 & 0 \end{bmatrix}.$$

det(A) = −2 ⇒ A is invertible

$$A^{-1} = -\frac{1}{2}\begin{bmatrix} 0 & -1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} \\ 1 & \frac{3}{2} \end{bmatrix}$$

(b)
$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix}.$$

det(A) = −6 ⇒ A is invertible

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & -3 & -2 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{2}{3} & 0 & -\frac{1}{3} \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{3} & 0 & \frac{2}{3} \\ 0 & 1 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{2}{3} & 0 & -\frac{1}{3} \end{array}\right]$$

$$\underbrace{\begin{bmatrix} -\frac{1}{3} & 0 & \frac{2}{3} \\ 0 & \frac{1}{2} & 0 \\ \frac{2}{3} & 0 & -\frac{1}{3} \end{bmatrix}}_{=A^{-1}}$$

(c)
$$A = \begin{array}{c} \begin{array}{cccc} u_1 & u_2 & u_3 & u_4 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \end{array}.$$

$u_1 = -u_4$ ⇒ Columns of A are linearly dependent

⇒ det (A) = 0

⇒ A is not invertible

(6 points) 2. Let S be the parallelogram with sides given by the vectors

$$\begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

(a) Find the area of S. area of $S = \left|\det\begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}\right| = 5$

(b) Find the area of $T_A(S)$, where $T_A : \mathbb{R}^2 \to \mathbb{R}^2$ is the linear transformation defined by the matrix

$$A = \begin{bmatrix} -1 & 2 \\ 5 & 1 \end{bmatrix}.$$

$$A\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 16 \end{bmatrix} \quad \& \quad A\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -9 \end{bmatrix}$$

the area $= \left|\det\begin{bmatrix} -1 & 4 \\ 16 & -9 \end{bmatrix}\right| = |9-64| = 55$

$$A \to \begin{bmatrix} 1 & 0 & 1 & -1 & 3 \\ -3 & 1 & 0 & -1 & 2 \\ 1 & 0 & 1 & -1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & -1 & 3 \\ 0 & 1 & 3 & -4 & 11 \\ 0 & 0 & 0 & 0 & -3 \end{bmatrix} \rightsquigarrow \text{a basis for Column Space of } A = \left\{ \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} \right\}$$

$$x_1 = -x_3 + x_4$$
$$x_2 = -3x_3 + 4x_4 \quad \Rightarrow x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -x_3 + x_4 \\ -3x_3 + 4x_4 \\ x_3 \\ x_4 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 4 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$
$$x_5 = 0$$

(9 points) 3. For the matrix

$$A = \begin{bmatrix} -3 & 1 & 0 & -1 & 2 \\ 1 & 0 & 1 & -1 & 3 \\ 1 & 0 & 1 & -1 & 0 \end{bmatrix}$$

So, a basis for the null space $= \left\{ \begin{bmatrix} -1 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

answer the following questions.

(a) Find a basis for the column space of A.

(b) Find a basis for the null space of A.

(c) What is the rank and what is the nullity of A. rank $= 3$, nullity $= 2$

(6 points) 4. Assume A is a 3×3 matrix with $A^{-1} = B$, where

$$B = \begin{bmatrix} -3 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

(a) Find the determinant of A $\quad \det(B) = 3 - 1 + 0 = 2 \Rightarrow \det(A) = \frac{1}{\det(A^{-1})} = \frac{1}{\det(B)} = \frac{1}{2}$

(b) Find a solution to $Ax = y$ where

$$Ax = y \Rightarrow A^{-1}Ax = A^{-1}y \Rightarrow x = A^{-1}y = By$$

$$y = \begin{bmatrix} -2 \\ 3 \\ 6 \end{bmatrix}$$

$$\Rightarrow x = \begin{bmatrix} -3 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 9 \\ 4 \\ 13 \end{bmatrix}$$

(10 points) 5. Answer true or false for $n \times n$ matrices A and B. Justify your answers.

(a) If the column vectors of a matrix A are linearly independent, then the determinant of A is not zero.

(b) The matrix A can be one-to-one, but not onto.

(c) The number of pivots of A equals the nullity of A.

(d) It can never happen that $AB = BA$.

(e) It can happen that A is not invertible, but AB is invertible.

(a) True, if the column vectors of a matrix A are linearly independent, then A is invertible and therefore $\det(A) \neq 0$

(b) False, if A is 1-1, then # pivots = # Columns, and since # Columns = # rows = n, so # Pivots = # rows, and then A is onto as well.

(c) False, # pivots of A = rank of A & # free variables = nullity of A

(d) False, it is enough to take $A = B$, then $AB = A^2 = BA$

(e) False, if A is not invertible, then $\det(A) = 0$

So, $\det(AB) = \det(A)\det(B) = 0 \cdot \det(B) = 0 \Rightarrow AB$ is not invertible