Complexity of Knots and Integers
FAU Math Day

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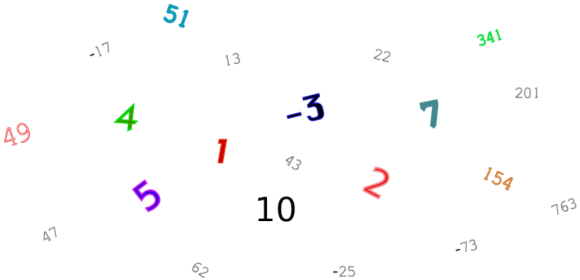
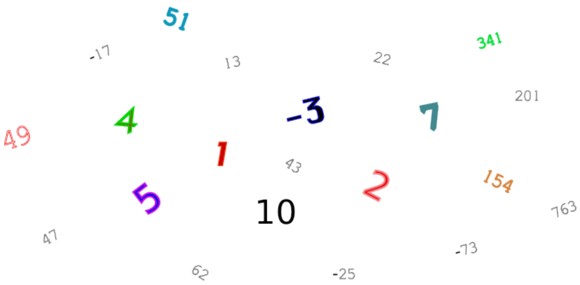
Part I: Lehmer's question

Integers



Integers

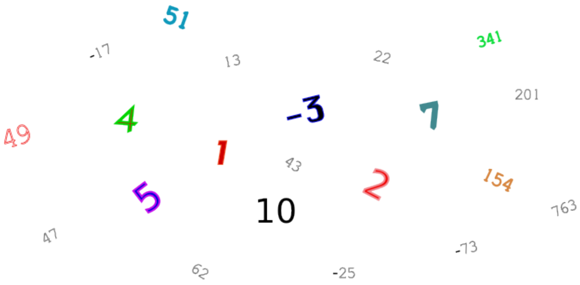


Properties:

- Ordering (total ordering)

$$\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots, 10, \ldots$$

Integers

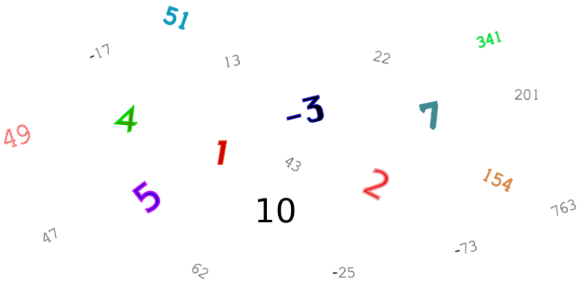


Properties:

- Size (absolute value)

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$$

Integers



Properties:

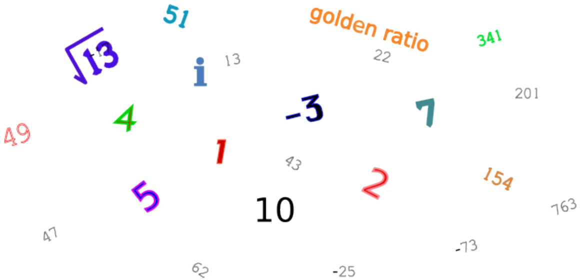
- Addition, subtraction, multiplication, division, powers

  e.g.

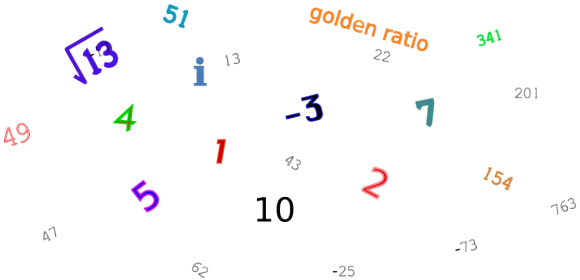
$$(17)^3 - 21(17)^2 + 70(17) - 34 = 0$$

or 17 is a root of the polynomial $p(x) = x^3 - 21x^2 + 70x - 34$.

Algebraic integers



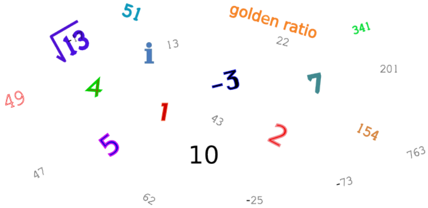
Algebraic integers



Definition: An *algebraic integer of degree d* is a root of a monic integer polynomial of degree d:

$$p(x) = x^d + a_{d-1}x^{d-1} + \cdots + a_0, \qquad a_i \in \mathbb{Z}.$$

Algebraic integers



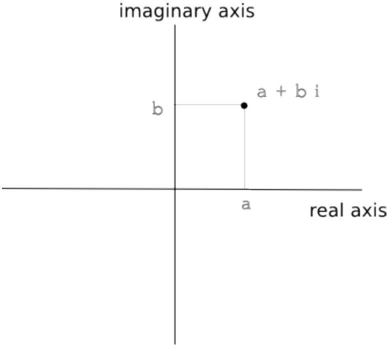
Includes:

- **integers**: e.g., $p(x) = x - 5$ has root 5.
- **roots of integers**: e.g., $p(x) = x^3 - 2$ has root equal to the cube root of 2.
- **complex numbers**: e.g., $p(x) = x^2 + 1$ has root equal to **i** .

Doesn't include *rational numbers* such as $\frac{1}{2}$ or **transcendental numbers** such as e and π.

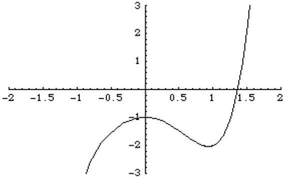
Theorem (Fundamental Theorem of Algebra.)

Every polynomial of degree d has d roots (counted with multiplicity) of the form $a + b\ i$ where a, b are real numbers, and $i = \sqrt{-1}$.

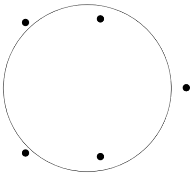


Degree 5 example

Graph of $p(x) = x^5 - x^2 - 1$ as a real function.



Roots of $p(x) = x^5 - x^2 - 1$ and unit circle in the complex plane.



Golden ratio

Golden ratio: $\phi = 1 + \frac{1}{1+\frac{1}{1+\frac{1}{1+\cdots}}} = 1.618034\ldots$

ϕ is a root of $x^2 - x - 1$.

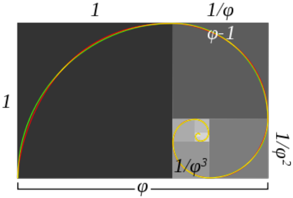


Figure: Acropolis in Athens, Greece, and spiraling squares

Golden mean and its conjugate

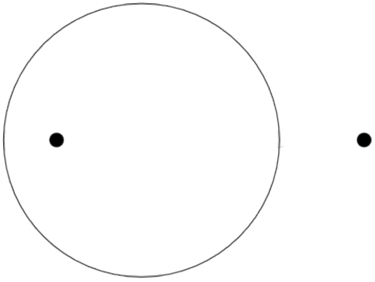
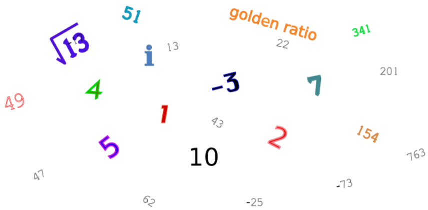


Figure: Roots of $p(x) = x^2 - x - 1$ and unit circle.

Algebraic integers



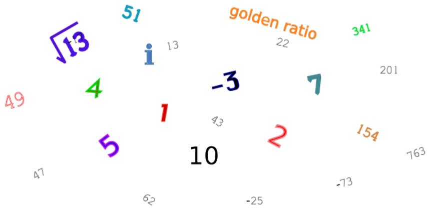
Size: (complex norm)

$$||a + b\ \mathbf{i}|| = \sqrt{a^2 + b^2}.$$

House of an algebraic integer: ($\alpha \sim \beta$ if they are *conjugate*, i.e., satisfy the same irreducible monic integer polynomial.)

$$H(\alpha) = \max\{||\beta|| \ : \ \alpha \sim \beta\}.$$

Algebraic integers



Ordering: What is the smallest integer greater than one?

- $\leq$ does not define a total ordering on $\mathbb{C}$.
- Houses are totally ordered, and they are dense in $[1, \infty)$.

Roots of unity.

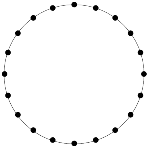


Figure: Roots of $p(x) = x^{20} - 1$.

A *root of unity* is a root of an equation

$$x^n = 1,$$

for some integer $n \geq 1$.
For algebraic integers: $H(\alpha) = 1 \Leftrightarrow \alpha$ is a root of unity.

Complexity of an algebraic integer

We measure the *compexity* of an algebraic integer in terms of the roots outside the unit circle.

- The number of roots outside the unit circle $N(p)$.
- The largest absolute value of a root of $p(x)$, called the *house* of p.
- The *Mahler measure* of a monic integer polynomial is the absolute value of the product of roots outside the unit circle:

$$\mathrm{Mah}(p(x)) = \Pi_{p(\mu)=0} \max\{|\mu|, 1\}.$$

Simplest case: the roots of $p(x)$ are roots of unity, i.e., $p(x)$ is *cyclotomic*.

- $N(p) = 0$
- House of $p(x)$ is 1
- The Mahler measure of p is 1.

Other extreme:

- $N(p)$ can be as large as the degree of $p(x)$:
- House of $p(x)$ can be arbitrarily close to 1.

Example: $p(x) = x^n - 2$.

Lehmer's Question[1]

The Mahler measure is more mysterious.

Question (Lehmer's question)

Is there a gap between 1 *and the next largest Mahler measure?*

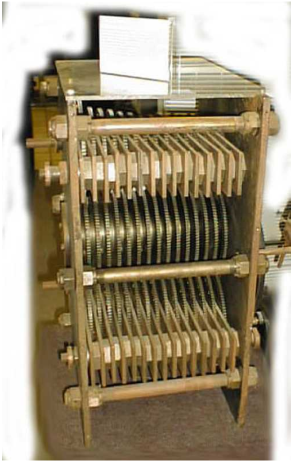
Relating the invariants...

$$H(\alpha) \leq M(\alpha) \leq H(\alpha)^{N(\alpha)}.$$

As $H(\alpha) \to 1$ does $N(\alpha) \to \infty$ correspondingly?

[1] *Factorization of Certain Cyclotomic Functions* by D.H. Lehmer, Annals of Math Studies, 2nd series, 1933 (pp. 461-479)

In addition to being a Berkeley Mathematics Professor, Lehmer was an inventor of computers...



From the golden mean to Lehmer's number

Heuristic for why there may be a gap.

Lehmer's list:

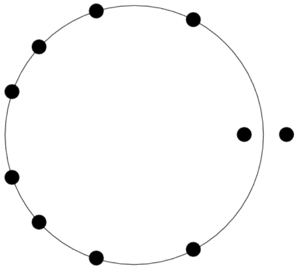
Palindromic polynomials with smallest Mahler measure of degrees ≤ 10.

$$
\begin{array}{rl}
x^2 - 3x + 1 & \lambda = (\text{golden mean})^2 \approx 2.618033 \\
x^4 - x^3 - x^2 - x + 1 & \mu \approx 1.7220838 \\
x^6 - x^4 - x^3 - x^2 + 1 & \mu \approx 1.4012683 \\
x^8 - x^5 - x^4 - x^3 + 1 & \mu \approx 1.2806381 \\
\text{Lehmer's polynomial} & \mu = \text{Lehmer's number} \approx 1.17628082
\end{array}
$$

After this, smallest Mahler measures start to go up again (though not monotonically).

Roots of Lehmer's polynomial

$p_L(x) = x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1.$



Lehmer's polynomial is also a *Salem* polynomial.

Polynomials with small Mahler measure

- **Non-reciprocal case:**

  (Smyth '70) The smallest Mahler measure for non-reciprocal polynomials is given by

  $$\mu_S = \mathrm{Mah}(x^3 - x - 1) \approx 1.32472,$$

  $x^3 - x - 1$ is a Pisot Polynomial (only one root outside the unit circle).

Polynomials with small Mahler measure

- **Conjectural smallest:**

  Lehmer's polyomial

  $$p_L(x) = x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1$$

  has the smallest known Mahler measure greater than one[2] :

  $$\mu_L \approx 1.17628082 < \mu_S.$$

It follows that to answer Lehmer's question, it suffices to study palindromic polynomials.

[2] This has been verified by Boyd and Mossinghoff up to degree 44. See http://www.cecm.sfu.ca/∼mjm/Lehmer/ for latest news on Lehmer's question and related topics.

Two families of polyomials

- Salem-Boyd polynomials
- Lanneau-Thiffeault polynomials

Salem-Boyd polynomials

R. Salem (1960s) and D. Boyd (1990s-2000s)

$$P_n(x) = x^n f(x) + f^*(x).$$

Properties:

- House(P_n) $\to$ House(f)
- $N(P_n) \to N(f)$ and
- Mahler measure(P_n) $\to$ Mahler measure(f).

Salem-Boyd polyomials

In particular, if we set $f(x) = x^3 - x - 1$, then

$$P_n(x) = x^n(x^3 - x - 1) + x^3 + x^2 - 1.$$

and we have

- $P_n(x)$ is cyclotomic for $n < 8$,
- all the $P_n(x)$ are Salem polynomials for $n \geq 8$, and
- Lehmer's polynomial is a factor of $P_8(x)$:

$$P_8(x) = (x - 1)\text{Lehmer's polynomial}$$

Consequence: Every Pisot number is a limit of Salem numbers.

Lanneau-Thiffeault polynomials

E. Lanneau and J.L. Thiffeault (2009)

$$LT_n(x) = x^{2n} - x^{n+1} - x^n - x^{n-1} + 1$$

- Lehmer's polynomial is a factor of $x^{12} - x^7 - x^6 - x^5 + 1$.
- The other three polynomials on Lehmer's list are also LT-polynomials.
- A $n \to \infty$,

$$H(LT_n(x))^n \longrightarrow \phi^2 = H(x^2 - 3x + 1).$$

In particular, $H(LT_n(x)) \to 1$ and the behavior is asymptotic to $\phi^{\frac{2}{n}}$.

General families of polynomials

Useful features:

- The coefficients stay the same, only the exponents change.
- The normalized house of the polynomials behave nicely as n varies.
- These kinds of families occur naturally in the setting of low-dimensional geometry and topology.

Part II: Knots and Links

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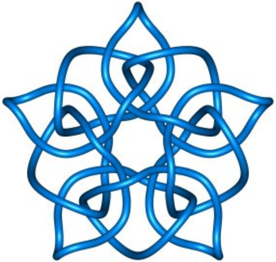


From http://knotplot.com.

Formal Definitions

A *knot* is a closed loop in the 3-dimensional sphere:

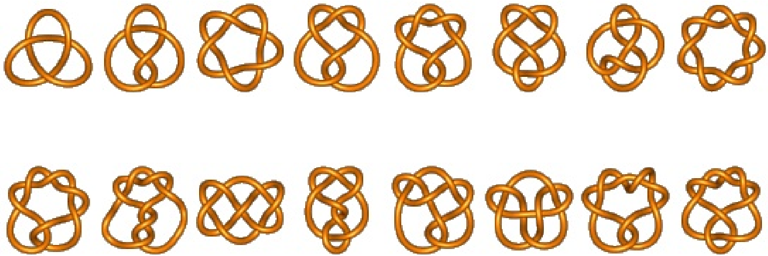
$$S^3 = \mathbb{R}^3 \cup \{\infty\}.$$

A *link* is a disjoint union of knots in the 3-dimensional sphere.

Two knots or links are considered to be the *same* if you can move one to the other without breaking the string.

Tables of knots[3]

Sixteen knots with fewest crossing numbers.



[3] http://www.knotplot.com/knot-theory/ by Robert G. Scharein (UBC PhD 1998) "There are 12,965 knots with 13 or fewer crossings in a minimal projection and 1,701,935 with 16 or fewer crossings."

Three dimensional manifolds

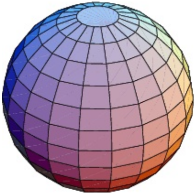
We can sometimes distinguish knots by their *exteriors* $S^3 \setminus K$.

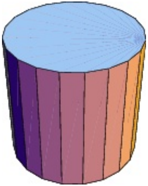
This is a 3-dimensional manifold (locally looks like 3-dimensional space). We can study the topology and geometry.

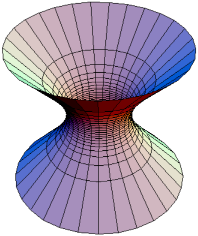
William Thurston's insight (Geometrization Conjecture): For 3-dimensional manifolds, once you know the topology, the geometry is determined. Any 3-dimensional manifold can be "cut into irreducible pieces" in a precise way so that each piece can be represented uniquely as one of 8 geometries. [4]

[4] This conjecture was settled in the affirmative by Grigori Perelman in 2003

Two dimensional geometries







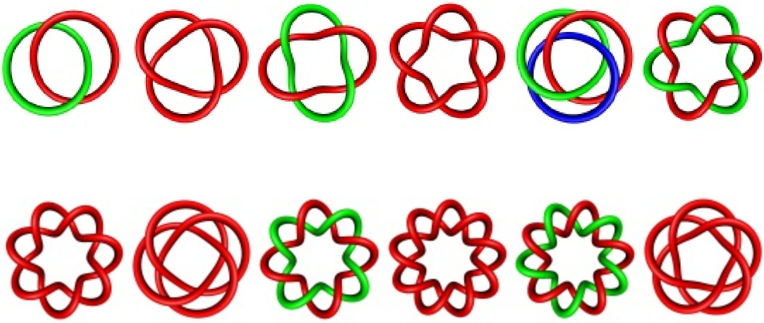
Knot and link complements

Geometry of a knot or link exterior. A knot or link is called *irreducible* if its complement is irreducible in the sense of Thurston. For irreducible knots and links K there are two possibilities: either K is a torus knot or link, or the complement is hyperbolic. (Thurston).

For a torus knot or link, the exterior is a product $\Sigma \times S^1$; the exterior can be foliated by parallel closed orbits.

In the hyperbolic case, the exterior is negatively curved. Straight lines can diverge.

Examples of torus knots



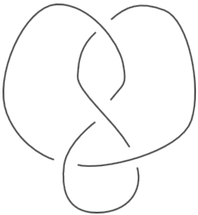
Alexander polynomials.

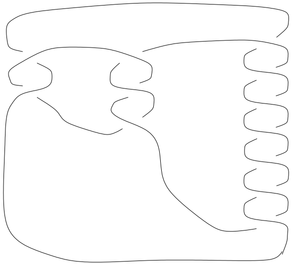
How can we measure the complexity of a hyperbolic knot or link?

A knot or link K has an associated polynomial Δ_K called the *Alexander polynomial*. These are polynomials in k variables, where k is the number components of K.

- If Δ_K is not cyclotomic, then K is hyperbolic.
- If K is a torus knot, then Δ_K is cyclotomic.
- Every monic reciprocal integer polynomial is realized as Δ_K for some fibered knot or link K.

Figure 8 and -2,3,7-pretzel knot





- Both are hyperbolic and fibered.
- Figure 8 has smallest volume among all knot complements.
- Both have lots of non-hyperbolic Dehn fillings (in this sense they are close to being non-hyperbolic).

Observations:

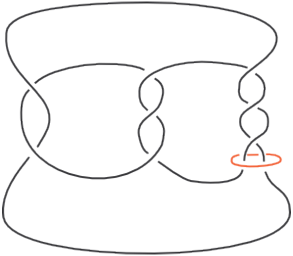
Knot	$H(\Delta)$	Δ
K_8	$(\text{golden mean})^2$	$\Delta_{K_8}(x) = x^2 - 3x + 1$
$K_{2,3,7}$	Lehmer's number	$\Delta_{K_{2,3,7}}(-x) =$ Lehmer's polynomial

- Can we relate "hyperbolic complexity" to invariants of Alexander polynomial?
- What can be said about Alexander polynomials associated to families of knots?

Families of knots determined by a link

Alexander polynomial

$$\Delta(u, t) = u(t^3 - t - 1) + t^3 + t^2 - 1.$$

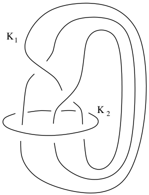


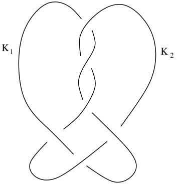
Salem-Boyd polynomials: $\Delta(x^n, x)$.

Families of knots determined by a link

Alexander polynomial

$$\Delta(u,t) = u^2 - u(1 - t - t^{-1}) + 1.$$





LT -polynomials: $\Delta(x^{2n}, -x) = LT_n(x)$.

Summary

- Multivariable polynomials organize algebraic integers into natural families.
- For example, LT-polynomials provide a way to interpolate between the first 4 polynomials on Lehmer's list.
- Geometric relations between knots and links give rise to families of associated polynomials.
- For example the figure eight knot and the -2,3,7 knot have a deep geometric connection, which is reflected in a connection between the golden mean and Lehmer's number via LT polynomials.

References

Factorization of Certain Cyclotomic Functions by D.H. Lehmer, Annals of Math Studies, 2nd series, 1933 (pp. 461-479)

http://www.cecm.sfu.ca/∼mjm/Lehmer/ website maintained by M. Mossinghoff

http://www.knotplot.com/knot-theory/ website maintained by R. G. Scharein

Thank you!