Two-dimensional topological fluid dynamics

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Colloquium

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Modeling fluids

- There are two interconnected parts of fluid modeling: Lagrangian which follows the fluid as its moves and Eulerian which sits at a point and considers local quantities like the velocity.
- Lagrangian:
  - The trajectory through space of a fluid particle beginning at position $\mathbf{x}$ is given by a function $\phi(t)$ with $\phi(0) = \mathbf{x}$ and $\phi(t)$ the position of the particle after time t.
  - If M is the fluid domain, all these trajectories are collected together in a single function, the fluid motion,

$$\phi : M \times \mathbb{R} \to M,$$

usually written $\phi_t(\mathbf{x}) = \phi(\mathbf{x}, t)$.

Modeling fluids

- Eulerian:
  - The velocity field is

$$\mathbf{u}(\phi_t(\mathbf{x}), t) := \frac{\partial \phi_t}{\partial t}(\mathbf{x}). \tag{1}$$

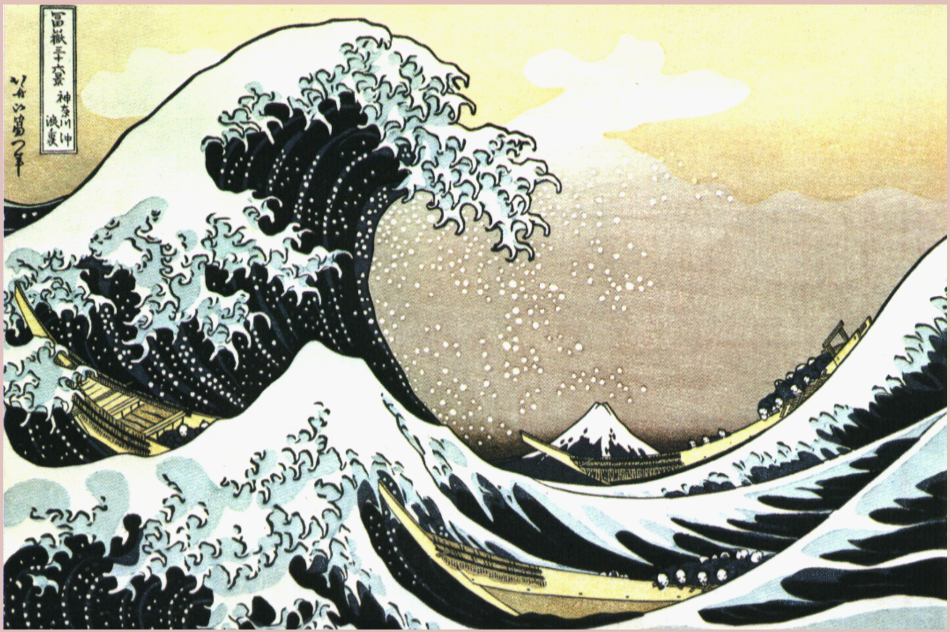
  - The equations of fluid mechanics are usually written using the velocity field. One then solves the advection equation (1) for the trajectories.
  - For example, the Navier-Stokes equation

$$\rho \frac{DX}{Dt} = -\nabla p_t + \nu \Delta X,$$

with appropriate boundary conditions, where ν is the viscosity and p_t is the pressure.

Deformation and vorticity

- When we watch a fluid evolve, there seem to be (at least) two fundamental things going on, stretching and rotating.
- These are expressed infinitesimally at each point (Eulerian) using the space derivative of the velocity field $\nabla \mathbf{u}$.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The deformation

- The symmetric part of $\nabla \mathbf{u}$ can be orthogonally diagonalized yielding the instantaneous, infinitesimal deformation

$$(\nabla \mathbf{u})_{sym} := \frac{\nabla \mathbf{u} + (\nabla \mathbf{u})^T}{2} \sim \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix}$$

- So again locally and instantaneously, $d\mathbf{x}/dt = (\nabla \mathbf{u})_{sym} \cdot \mathbf{x}$ integrates to trajectories

$$\Phi_t(\mathbf{x}) = \begin{pmatrix} e^{d_1 t} & 0 & 0 \\ 0 & e^{d_2 t} & 0 \\ 0 & 0 & e^{d_3 t} \end{pmatrix} \cdot \mathbf{x}$$

Vorticity

- The anti-symmetric part of $\nabla \mathbf{u}$ yields the instantaneous, infinitesimal curl or vorticity, $\vec{\omega} = (\omega_1, \omega_2, \omega_3)$,

$$(\nabla \mathbf{u})_{anti} := \frac{\nabla \mathbf{u} - (\nabla \mathbf{u})^T}{2} = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix},$$

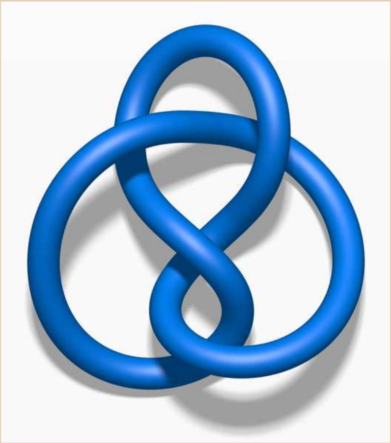
  and $(\nabla \mathbf{u})_{anti} \cdot \mathbf{x} = \vec{\omega} \times \mathbf{x}$

- So again locally and instantaneously, $d\mathbf{x}/dt = (\nabla \mathbf{u})_{sym} \cdot \mathbf{x}$ integrates to trajectories which rotate around the axis $\vec{\omega}$ with angular velocity $|\vec{\omega}|$.

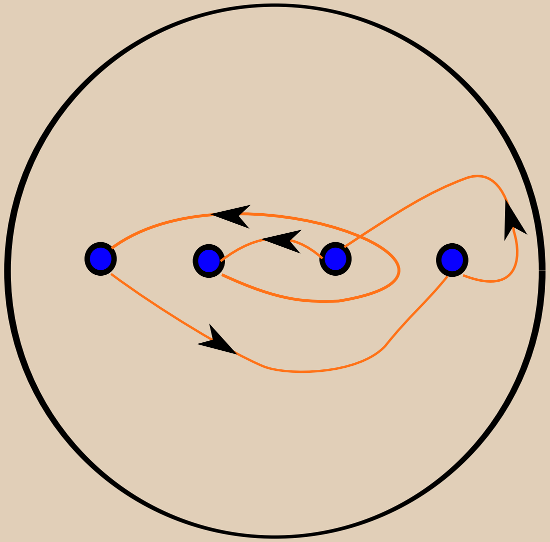
Back to modeling

- Solving the Navier-Stokes equation is very hard. One must then solve the advection equation (1) for the physical fluid trajectories.
- It is not clear at all how the Eulerian infinitesimal and instantaneous quantities of deformation and vorticity contribute to actual macroscopic Lagrangian deformations and rotations.
- Thus there are a host of analytic and qualitative methods for getting information about the fluid evolution without going through all these levels of analysis.
- One class of methods go under the name of Topological Fluid Mechanics which combines ideas from Topology and Dynamical Systems theory. Today's talk is about some of these methods in two-dimensions.

Knotting



- Knots are an essential ingredient of three-dimensional topology and thus of $3D$ fluid dynamics.
- What about two dimensions?
- Co-dimension two is necessary for knotting.



- Points are codimension two in the plane. Can they be knotted? Yes, if we consider the motion of the points.
- Question: what are the implications of knotted point motions?
- Answer: Exponential growth of material lines.

Experimental illustration

- Show Movie
- Very roughly, on the left (pA) there is pure deformation (the rotations cancel out) and on the right ((fo) there is pure rotation.
- On the left (pA) material lines are growing exponentially in length and and on the right ((fo) there is linear growth.
- The left clearly mixes better than the right.
- For a very viscous fluid the two protocols require very close to the same energy.
- Note the emergent structure on the left.
- Today's talk will focus on the exponential growth of material lines and its implications for deformation, vorticity, and mixing.

Outline

- Part 1: Kinematics and Topological Kinematics.
- Part 2: Consequences for passively transported scalars (eg. the cream in your coffee).
- Part 3: Applications to Euler flows.
- Part 4: Choosing protocols to maximize a topological measure of mixing efficiency.

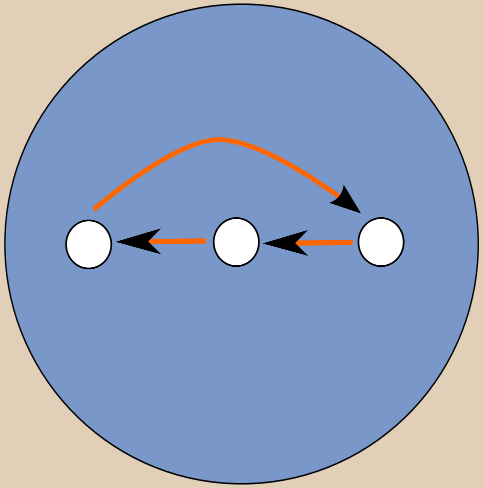
Main ideas:

- Topological one-dimensional growth of material lines is computable from knotted point motions.
- Topological growth is lower bound for metric growth.
- Metric one-dimensional growth of material lines is applicable to two-dimensional fluid mechanics.
  - Exponential growth of material lines implies exponential growth of gradients of passively transported scalars.
  - Large gradients enhance diffusion and thus mixing.
  - Interfaces between materials are one-dimensional curves and long interfaces also enhance mixing.
- Progression of ideas: Topology $\rightarrow$ Geometry $\rightarrow$ Analysis $\rightarrow$ Fluid Mechanics.

Part 1

Kinematics and topological kinematics

Basic definitions: The fluid region



The fluid region is a smooth, one-parameter family of smooth, multi-connected, compact, planar domains M_t.

- The outer boundary is held fixed while the inner disks move.
- Always assume time-periodicity, $M_{t+1} = M_t$, and thus model stirring by moving rods and use the terminology stirring protocol to describe the moving regions M_t.
- The moving regions are called the stirrers, and they are perhaps permuted each cycle.

Basic definitions: the fluid motion

- The fluid motion is a smooth one-parameter family of diffeomorphisms, $\phi_t : M_0 \to M_t$, with $\phi_0 = id$, just kinematics.
- View ϕ_t as Lagrangian fluid displacement map: particle at $\mathbf{x} \in M_0$ at time 0 is at $\phi_t(\mathbf{x}) \in M_t$ at time t.
- I am avoiding the terminology "flow" because in dynamical systems theory this means an $\mathbb{R}$-action, *ie.* a steady flow in fluid mechanics, which usually won't be the case here.
- The fluid motion is incompressible if it preserves Lebesgue measure or equivalently, $\det(D\phi_t) \equiv 1$.

Basic definitions: the velocity field

- The velocity field is

$$\mathbf{u}(\phi_t(\mathbf{x}), t) := \frac{\partial \phi_t}{\partial t}(\mathbf{x}).$$

- Since $\phi_t : M_0 \to M_t$ the velocity field satisfies the boundary conditions $\mathbf{u} \cdot \mathbf{n}_i = \dot{B}_i \cdot \mathbf{n}_i$, with B_i the motion of the i^{th} boundary.
- Initially strictly kinematics or dynamical systems and so the velocity field is not yet assumed to satisfy any particular equation.
- The stirring protocol is time-periodic but the velocity field is perhaps not and so there is no Poincaré map in general.
- The fluid motion is incompressible iff $\mathrm{div}\,\mathbf{u} \equiv 0$.

One-dimensional metric growth rate

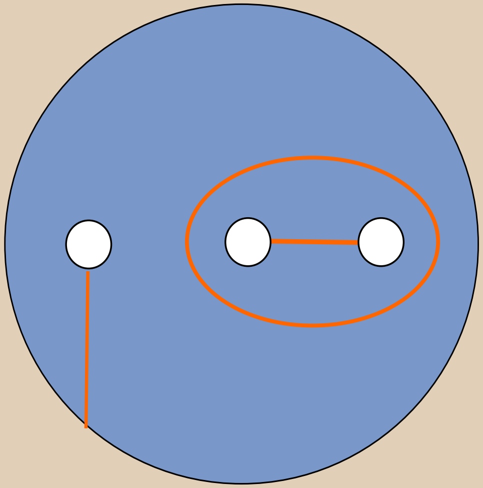
- A material line in the fluid is described by a smooth arc or simple closed curve (scc) γ. Let $\ell_t(\gamma)$ be its length with respect to some smooth, periodic family of Riemannian metrics on the M_t.
- The metric growth rate of γ is the growth of

$$L_t^{met}(\gamma) := \frac{\ell_t(\phi_t \circ \gamma)}{\ell_0(\gamma)}.$$

- On surfaces, the maximal exponential metric growth rate gives the topological entropy, connected to Lyapunov exponents, etc.

One-dimensional topological growth rate

- For the topological growth rate, we compute least length in a homotopy class, or equivalently, the length of an appropriate geodesic.
- To make the result strictly topological restrict consideration to just topologically essential curves.



An *essential arc* is one that connects two different boundary components. An *essential simple close curve (scc)* is one that is neither null-homotopic nor boundary parallel

One-dimensional topological growth rate

- The homotopy class of an essential arc allows the endpoints to slide along the boundary and for scc use free homotopy classes. In both cases the class is denoted $[\gamma]$.
- The least length among curves in $\gamma' s$ homotopy class is

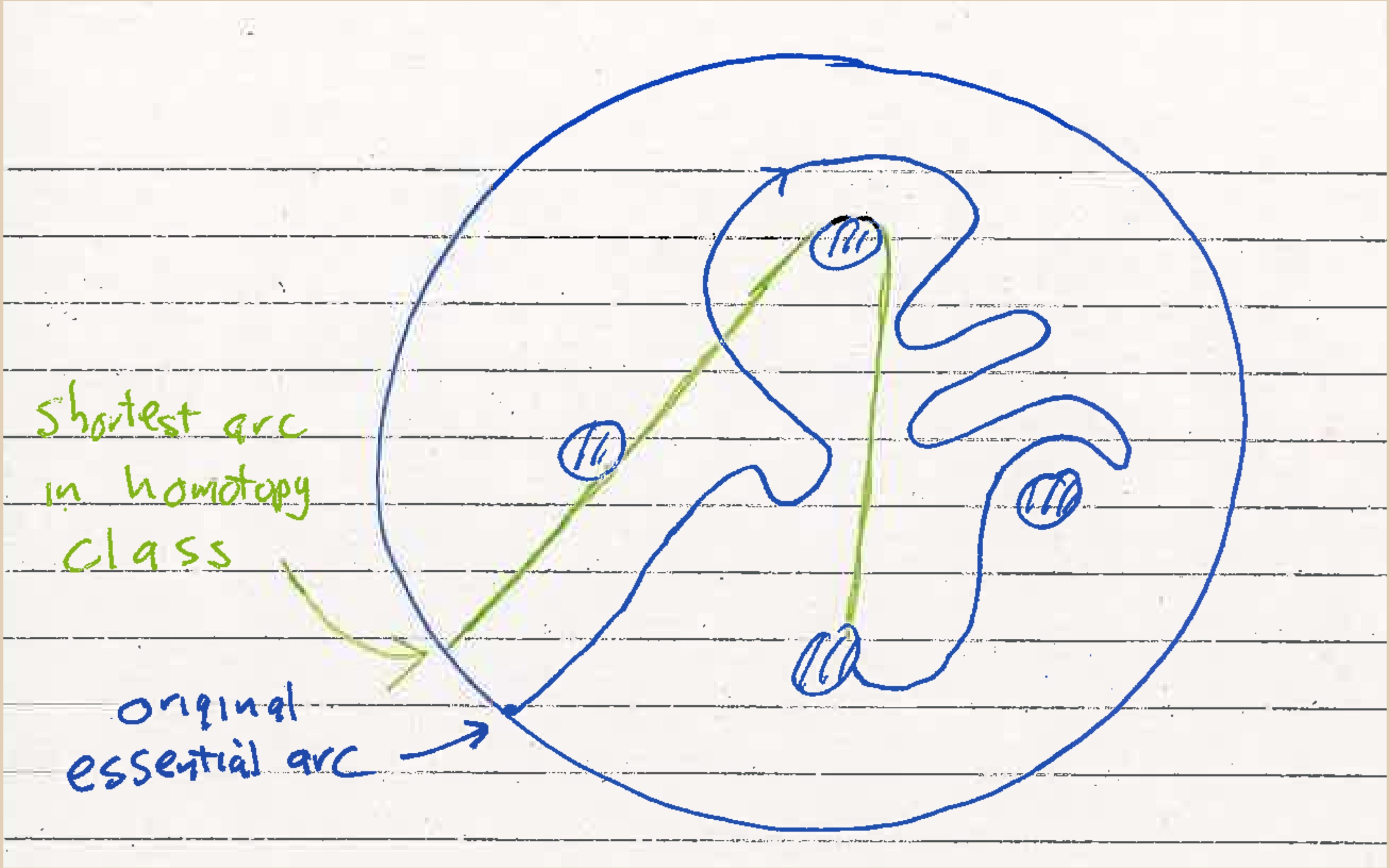
$$L^{top}(\gamma) := \min\{\ell(\sigma) \colon \sigma \in [\gamma]\}$$

- The topological growth rate of the class $[\gamma]$ is the growth of

$$L_t^{top}(\gamma) = \frac{L^{top}(\phi_t \circ \gamma)}{L^{top}(\gamma)},$$

- So we evolve curve forward for time t and then shrink to the least length in homotopy class.
- N.B. For an essential curve γ, $L_t^{met}(\gamma) \geq L_t^{top}(\gamma)$.

The least length in a homotopy class



One-dimensional topological growth rate

- The topological growth rate only depends on the rough topology of the stirrer motion.
- More precisely, recall two homeomorphisms $f_0, f_1 : M_0 \to M_0$ are isotopic if there is a continuous family of homeomorphisms f_t deforming one to the other.
- The topological growth rate of an essential curve γ depends only on the isotopy class of ϕ_1 (since the protocol is periodic, this is the same isotopy class as ϕ_n for all $n \in \mathbb{N}$).
- The topological growth is the same as the growth rate of word length for the induced map on the fundamental group (Cayley graph of fundamental group is quasi-isometric to the universal cover).

Thurston-Nielsen theory

- In the language of this talk, the Thurston-Nielsen theory classifies surface maps and their isotopy classes in terms of the rate of topological one-dimensional growth, linear or exponential, and give methods for computing the growth associated with specific protocols.
- The full theory deals with isotopy (mapping classes) on any surface.

The Thurston-Nielsen trichotomy

Let M_t be periodic stirring protocol with fluid motion ϕ_t. Then either

1. PseudoAnosov (pA): there exist constants $\lambda > 1$ (the dilation) and $0 < C_1 < C_2$ such that for every essential curve γ,

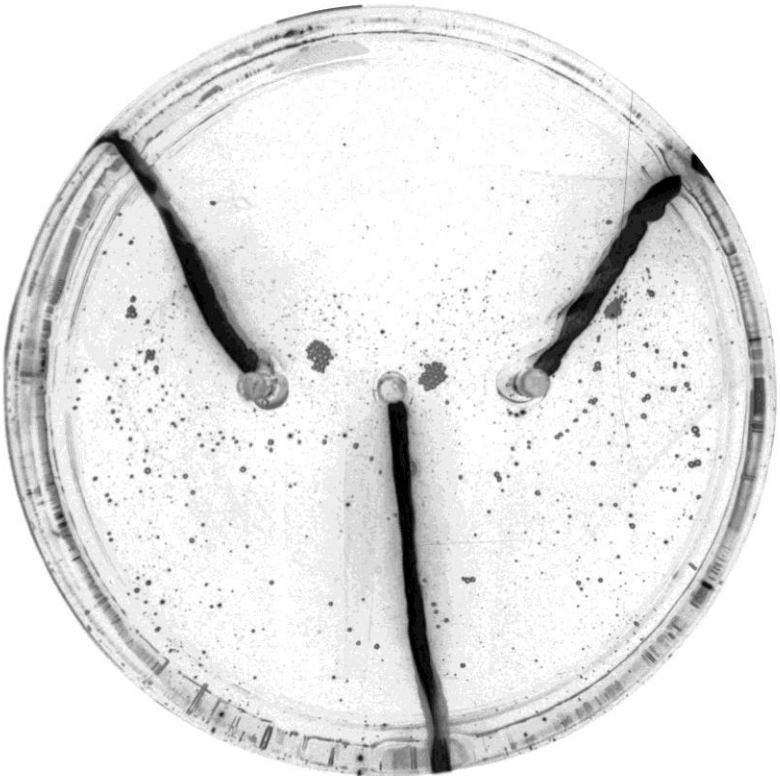
$$C_1 \lambda^t \leq L_t^{top}(\gamma) \leq C_2 \lambda^t.$$

2. Finite order (fo): there exists a constant $K > 0$ such that for every essential curve γ,

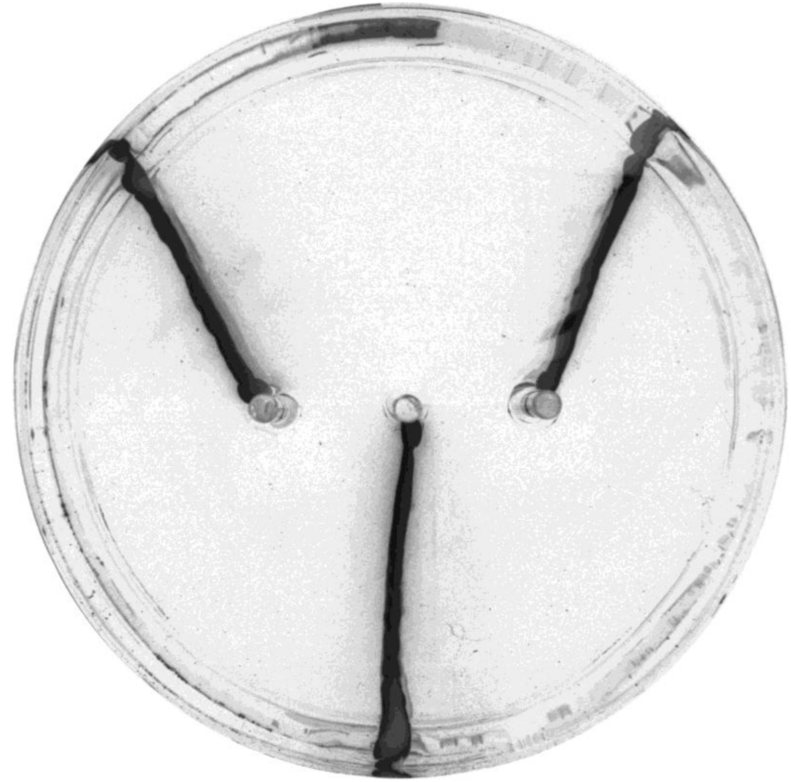
$$L_t^{top}(\gamma) < K\, n.$$

3. Reducible case: (roughly stated) M_0 splits into ϕ_1-invariant subsurfaces on which (1) or (2) holds.

Initial state



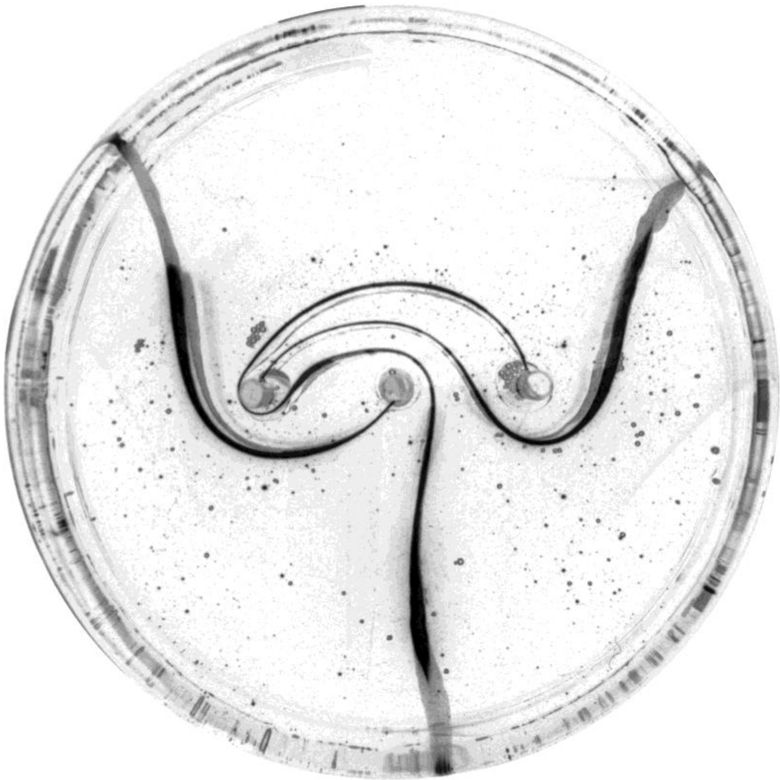
PseudoAnosov

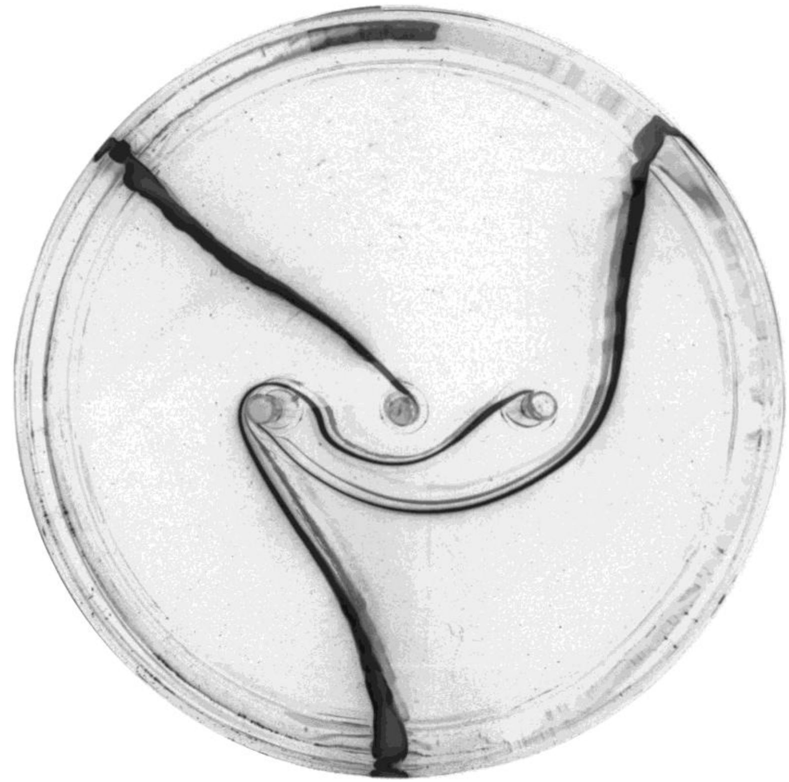


Finite Order

Experiment by Mark Stremler, see Boyland, P., Aref, H. and Stremler, M., Topological fluid mechanics of stirring, *J. Fluid Mech*., **403**, 277--304, 2000.

1 iterate





PseudoAnosov

Finite Order

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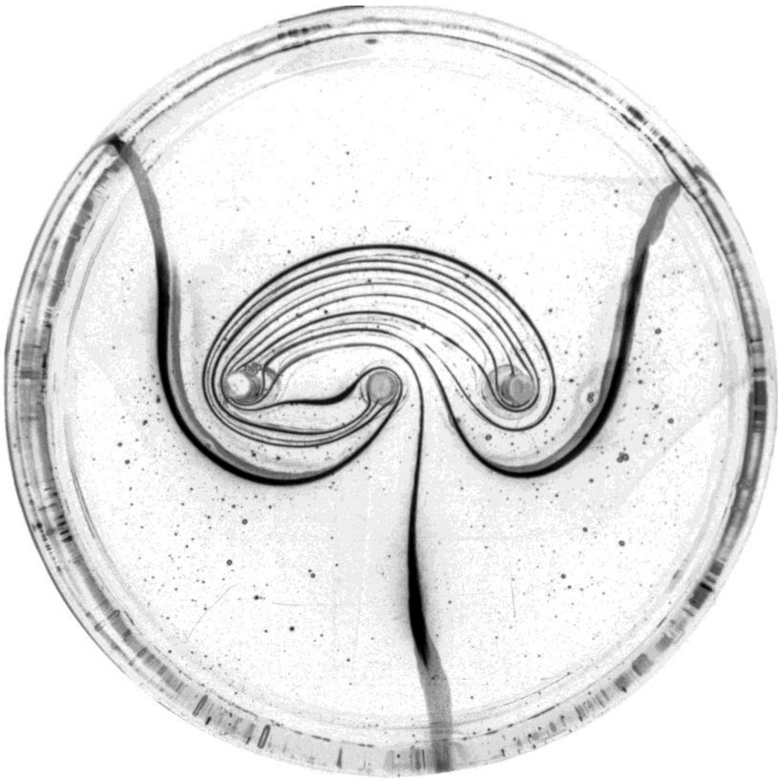
2 iterates

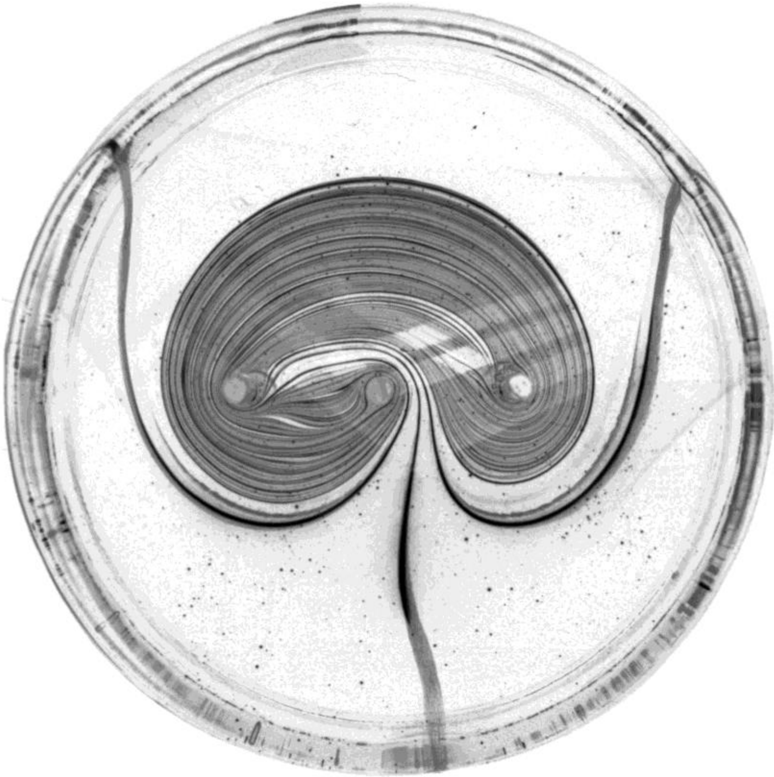


PseudoAnosov

Finite Order

Experiment by Mark Stremler, see Boyland, P., Aref, H. and Stremler, M., Topological fluid mechanics of stirring, *J. Fluid Mech*., **403**, 277--304, 2000.

9 iterates



PseudoAnosov

Finite Order

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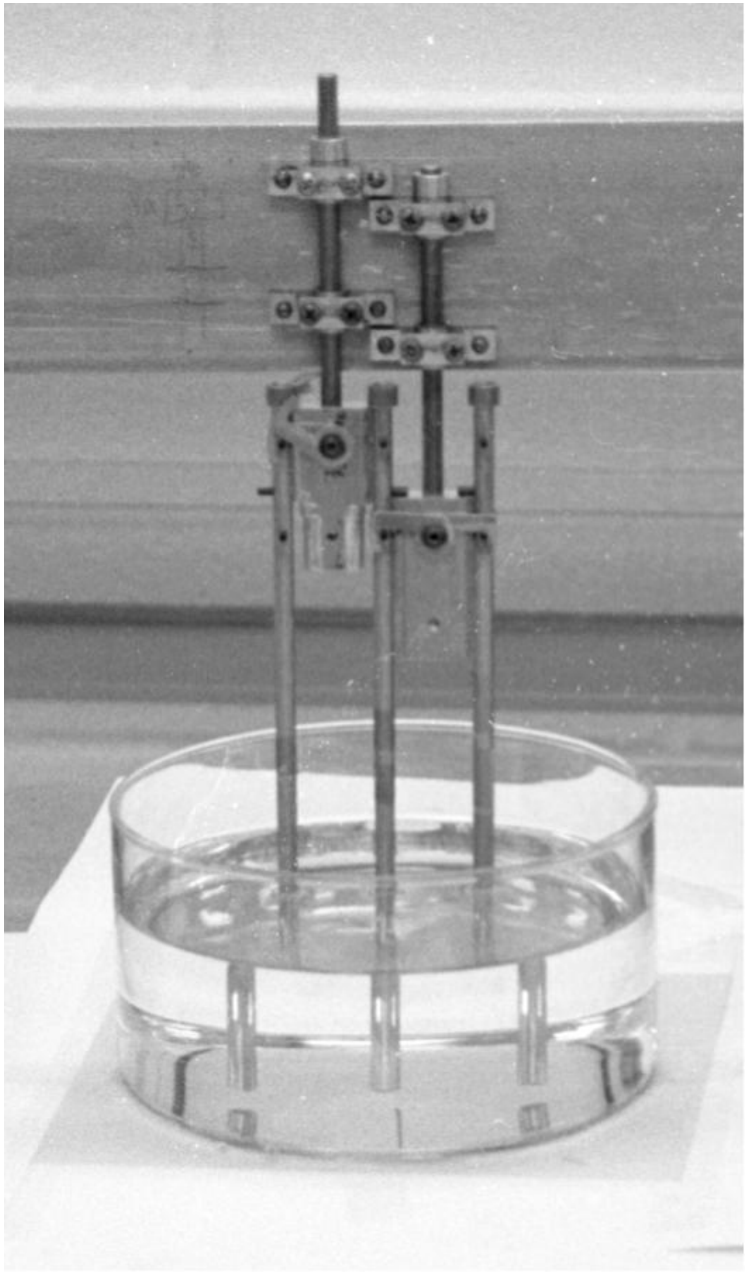


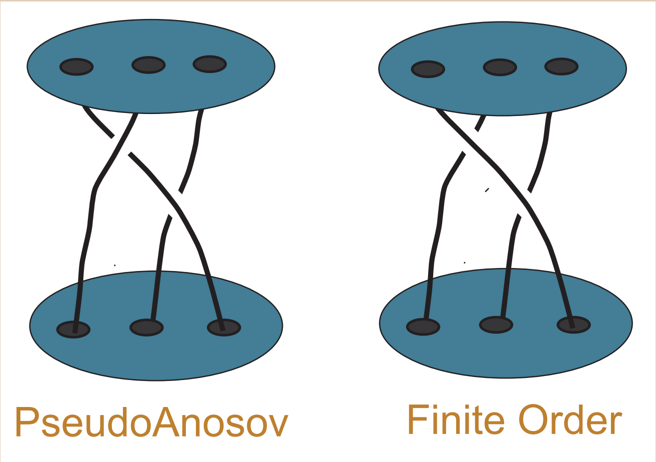
Remarks on TN trichomtomy

- We call the stirring protocol finite order, pseudoAnosov, or reducible according to the TN-type.
- We focus here just on the pseudoAnosov case, where every essential curve has the same topological exponential growth rate, namely, λ.
- This is independent of the details of the fluid motion, but just depends on the topology of the stirrer motion as described shortly.
- The topological growth is just a lower bound, the metric growth could be much larger.
- In the pseudoAnosov case the theory gives much more information about the dynamics: ways to compute the dilatation λ, periodic orbits and invariant measures that must be present, a lower bound of $\log(\lambda)$ for the topological entropy, etc.

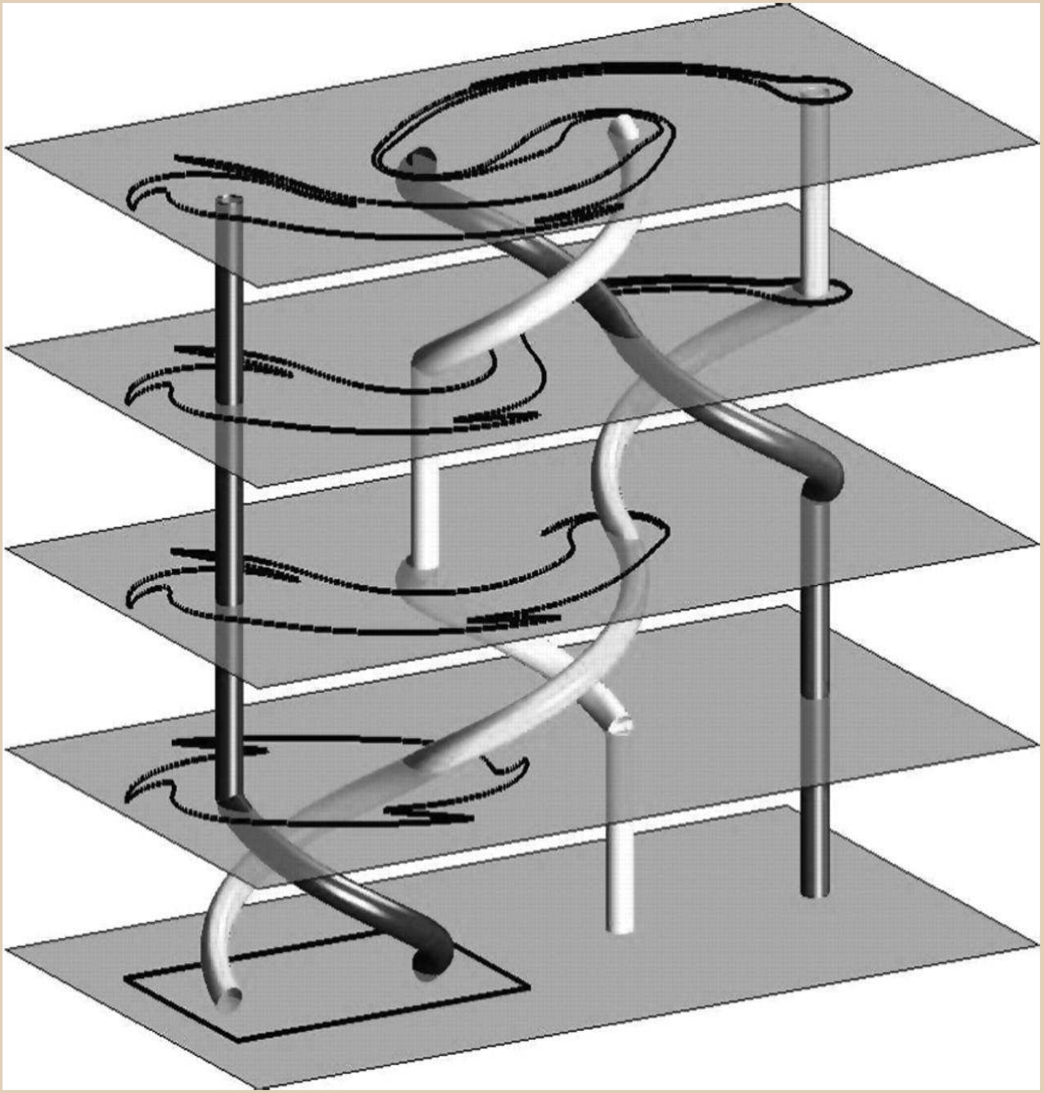
Braids, stirring protocols and isotopy classes

- TN type and the topological one-dimensional growth depend just on the istotopy class of ϕ_1.
- The isotopy class just depends on the topology of the motion of the stirrers and this in turn can be visualized and characterized by their space-time trace or braid.
- The algebra of the braid can be used to compute the TN-type.

The two protocols of the experiment have inequivalent braids; one is finite order (linear growth) and the other pA (exponential growth).



Braids and stirring protocols



Taken from Finn and Thiffeault.

Part 2

Passively advected scalars

Basic definitions

- Given a fluid motion ϕ_t, a function $\alpha : M_t \times \mathbb{R} \to \mathbb{R}$ is called a passively advected scalar if it is constant on trajectories,

  $$\alpha_t(\phi_t(\mathbf{x})) = \alpha_0(\mathbf{x}),$$

  or equivalently,

  $$i\frac{\partial \alpha_t(\phi_t(x))}{\partial t} = 0,$$

  where we have written $\alpha_t(\mathbf{x})$ for $\alpha(\mathbf{x}, t)$.
- Examples, dye in fluid, sugar in chocolate, or cream in coffee ignoring diffusion.
- In the language of global analysis one says that α_t is the push forward of α_0 and writes $(\phi_t)_*(\alpha_0) = \alpha_t$, with $(\phi_t)_*(\alpha_0) = \alpha_0 \circ (\phi_t)^{-1}$.

Two fundamental types of advected scalars

- For any function $f : M_0 \to \mathbb{R}$ we obtain a passively advected scalar just by defining $\alpha_t := (\phi_t)_*(f)$, and so only the initial configuration and the fluid motion matter.
- For example, if the density of a dye tracer is initially given by α_0 after time t the density is given by $\alpha_t := (\phi_t)_*(\alpha_0)$
- However, sometimes in a physical fluid α_t may represent a scalar of interest that is computed at each time from the velocity field. Thus is represents a conserved quantity.
- For example, in two dimensions the curl, $\omega_t = \nabla \times \mathbf{u}$, is a passively advected scalar for an Euler flow.
- The first case is relevant to mixing while the second to understanding dynamics of Euler fluids.

Heuristic consequences of a pseudoAnosov protocol

- For pseudoAnosov stirring protocol the metric length of essential curves is growing exponentially fast.
- Thus tangent vectors to these curves (material line elements) must be growing in length exponentially under the action of the space derivative of the fluid motion, $D\phi_t$.
- For a passively advected scalar α_t, since $\alpha_t = \alpha_0 \circ (\phi_t)^{-1}$,

$$\nabla \alpha_t = \nabla \alpha_0 (D\phi_t)^{-1}.$$

- If fluid motion is incompressible, $\det(D\phi_t) = 1$, and so $(D\phi_t)^{-1}$ also has an eigenvalue growing exponentially.
- Thus $|\nabla \alpha_t|$ is growing exponentially.

Issues with the argument

- There are (at least) three problems with making this rigourous.
  - The position along the material line at which we have growth of tangent vectors is moving in time
  - There could be an unfortunate coincidence where $\nabla\alpha_0$ stays aligned with the stable eigen-direction of $(D\phi_t)^{-1}$.
  - The scalar could have patches where $\nabla\alpha_0 \equiv 0$ and these could be just where the material lines are stretching.
- These issues are dealt with by using a more global argument and assuming the initial configuration of the scalar is generic in a precise sense.

Theorem on passively advected scalars

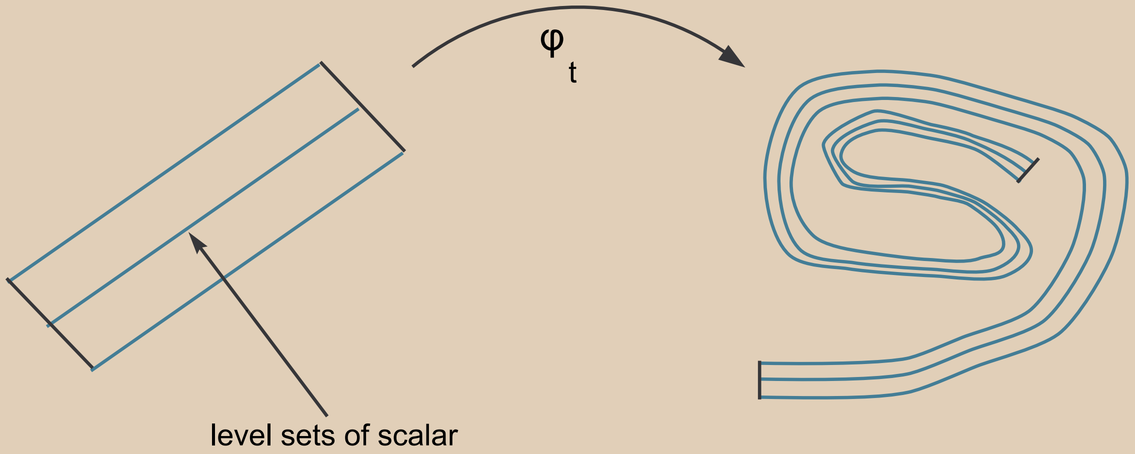
Theorem: M_t is a time-periodic stirring protocol of pA type with incompressible fluid motion ϕ_t. If α_t is a passively advected scalar such that its initial state α_0 is a generic C^2-function, then there are positive constants c, c' so that

$$\sup_{\mathbf{x} \in M_0} |\nabla \alpha_t(\mathbf{x})| \geq c\lambda^t \quad \text{and} \quad \int_{M_t} |\nabla \alpha_t(\mathbf{x})| \geq c' \lambda^t$$

for all $t \in \mathbb{R}$, where $\lambda > 1$ is the dilation of the pseudoAnosov protocol.

Thus $\|\alpha_t\|_{C^1}$ and $\|\alpha_t\|_{1,1}$ both go to infinity exponentially fast.

Idea of proof



- Find a C^2-open, dense set $\mathcal{G}$ inside the Morse functions on M_0 so that $\alpha_0 \in \mathcal{G}$ implies that α_0 has a band of regular inverse images of essential arcs or circles.
- The pA protocol forces stretch in length by λ^t. This coupled with area preservation and the passive transport force level sets of $\alpha_t = (\varphi_t)_*(\alpha_0)$ to bunch up in transverse direction, which causes $\|\nabla \alpha_t\|_\infty \to \infty$ like λ^t.

Note on time-periodic fluid motions

- If the velocity field is time-periodic with period one, then there is a Poincaré map ϕ_1 which satisfies $\phi_n = (\phi_{)}^{n}$ where a superscript is repeated composition.
- If there is a passively advected scalar α_t that depends on the velocity field, then $\alpha_{t+1} = \alpha_t$.
- This means that the scalar is an integral of motion

$$\alpha_0(\mathbf{x}) = \alpha_1(\phi_1(\mathbf{x})) = \alpha_0(\phi_1(\mathbf{x})).$$

- If α_0 is non-degenerate enough (eg. C^2-generic again), this implies that the dynamics of ϕ_t are very simple, in particular, it has zero topological entropy and at most linear one-dimensional metric growth of arcs.

Part 3
Applications to Euler Fluid Motions

Euler fluid motions

- Now assume the velocity field $\mathbf{u}(\mathbf{x}, t)$ of the fluid motion ϕ_t satisfies the incompressible, constant density ($\rho \equiv 1$), Euler equation
$$\frac{D\mathbf{u}}{Dt} = -\nabla p_t, \qquad \mathrm{div}(\mathbf{u}) = 0,$$
with slip boundary conditions on the moving boundary.
- Then ϕ_t is called an Euler fluid motion.
- Recall that for two-dimensional, divergence-free velocity fields a classical result says that the curl coupled with the circulations around the boundary components and the boundary conditions $\mathbf{u} \cdot \mathbf{n}_i = \dot{B}_i \cdot \mathbf{n}_i$ determine the field completely.

Existence of solutions: 2D Euler

- One has global classical solutions of the incompressible Euler equations in this case:
- **Theorem** *(Kozonoi 1985)* Given a smooth family of smooth compact planar regions M_t and any smooth divergence-free vector field $\mathbf{u}_0$ with slip boundary conditions $\mathbf{u}_0 \cdot \mathbf{n}_i = \dot{B}_i \cdot \mathbf{n}_i$ (or equivalently initial curl and circulations) on M_0, there is a unique, smooth Euler fluid motion with that initial data.
- In general, we assume a global solution with the regularity of the initial data and analyze its dynamics.

Helmholtz-Kelvin theorem: Euler fluid motions

- The Helmholtz-Kelvin Theorem (1890's) allows one to use the methods of dynamics/global analysis on Euler fluid motions.
- Helmholtz-Kelvin Theorem: A two-dimensional area-preserving fluid motion (M_t, φ_t) is Euler if and only if its vorticity is passively transported,

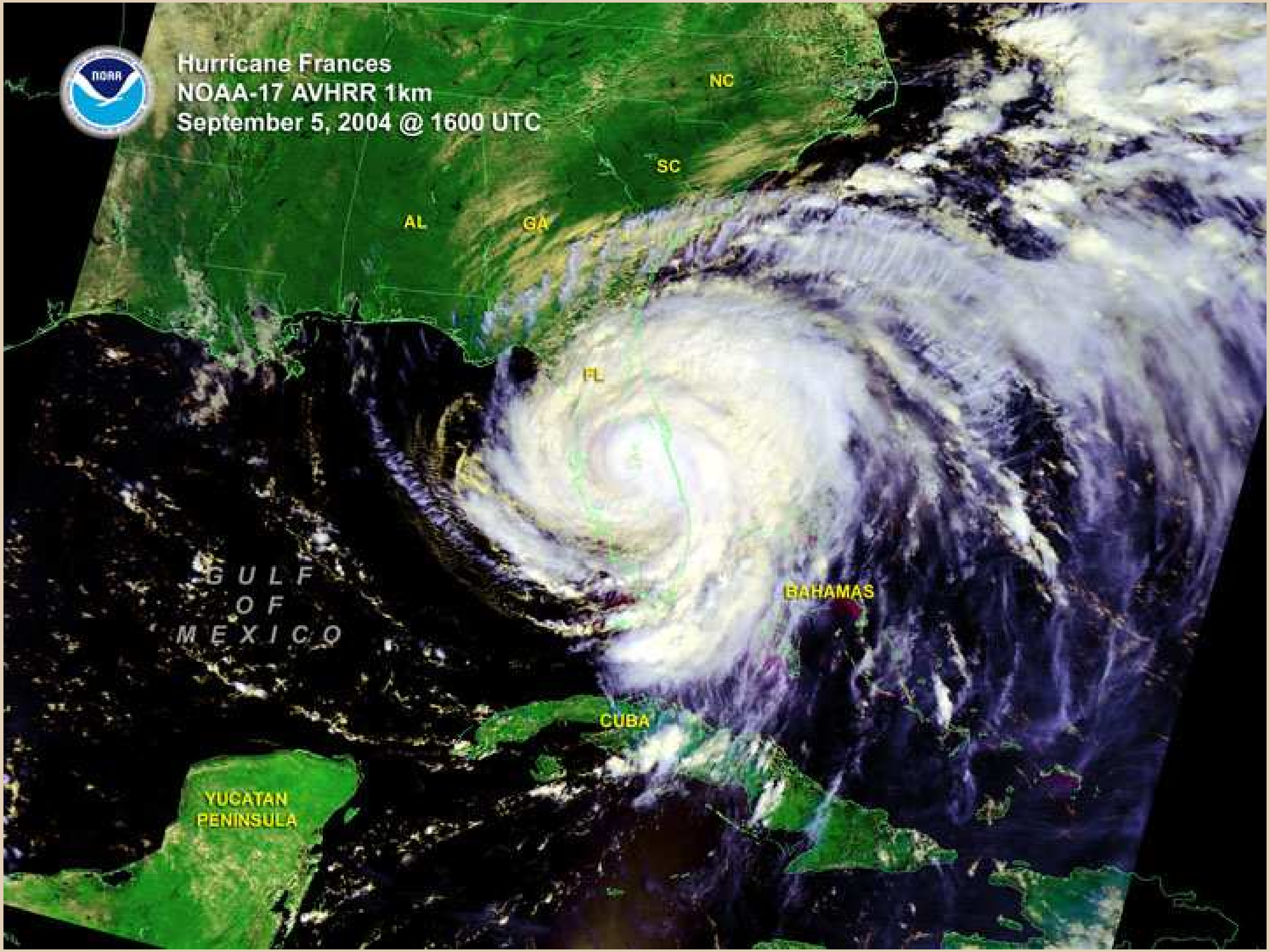
$$\frac{\partial \omega_t(\phi_t(x))}{\partial t} = 0$$

and circulations around all smooth simple closed curves C are preserved,

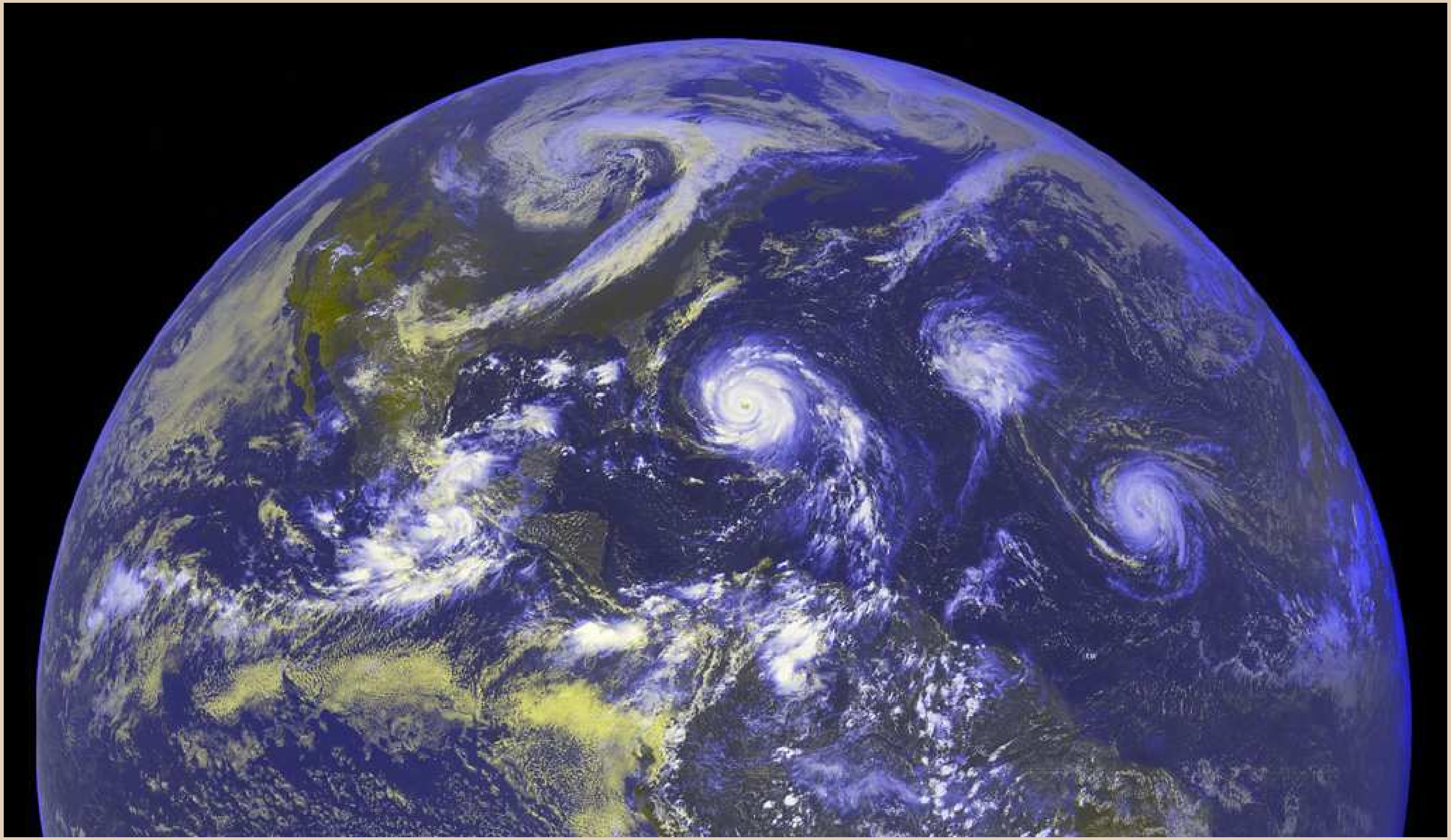
$$\frac{d}{dt} \oint_{\phi_t(C)} \mathbf{u} \cdot d\mathbf{r} = 0.$$

- The preservation of boundary circulation is a feature of multi-connected domains.

Transport of vorticity



Transport of vorticity



An exponential growth theorem

- Helmholtz-Kelvin says vorticity is a passively advected scalar for an Euler fluid motion and so we may use the theorem above:
- Theorem: Let M_t be a time-periodic stirring protocol of pA type with Euler fluid motion ϕ_t. If the initial vorticity ω_0 is a generic C^2-function, there are positive constants c, c' so that

$$\sup_{\mathbf{x}\in M_0} \|\nabla\omega_t(\mathbf{x})\| \geq c\lambda^t \quad \text{and} \quad \int_{M_t} \|\nabla\omega_t(\mathbf{x})\| \geq c'\lambda^t$$

  for all $t \in \mathbb{R}$ where $\lambda > 1$ is the dilation of the pA protocol.
- Thus $\|\Delta\mathbf{u}(\mathbf{x}, t)\|_\infty = \|\nabla\omega_t\|_\infty \to \infty$, $\|\mathbf{u}_t\|_{C^2} \to \infty$, and $\|\mathbf{u}_t\|_{2,1} \to \infty$ all like λ^t so $\mathbf{u}$ is not time-periodic.

Remarks on exponential growth theorem

- Yudovich (1974, 2000) and others showed linear growth of $\|\nabla\omega_t\|$ for perturbations of many two-dimensional steady Euler fluid motions. These results are central to stability analysis.
- Arnol'd (1972), Friedlander and Vishik (1992) and others have shown the importance of exponential growth of distortion for stability analysis.
- The basic mechanism in play here is the same: unbounded distortion as $t \to \infty$. Since $\omega_t = \omega_0 \circ \phi_t^{-1}$,

$$\nabla\omega_t = \nabla\omega_0 (D_{\mathbf{x}}\phi_t)^{-1}$$

- Here the growth of $\|\nabla\omega_t\|$ is exponential and forced by the topology of the pA stirring protocol which forces the maximal spectral radius of $D_{\mathbf{x}}\phi_t$ to grow like λ^t.

The energy of a stirred Euler fluid motion

- Total energy is conserved for Euler fluid motions in stationary bounded domains. What happens with moving boundary? Since $\|\nabla\omega\| \to \infty$, perhaps the energy is also unbounded?
- Usual argument for a steady domain yields

$$\frac{dE}{dt} = -\sum \oint_{\phi_t(C_i)} p\,\dot{B}_i \cdot d\mathbf{n}_i.$$

  since fluid can do work on the stirrers and vice versa (with the sum over the boundary circles).
- However, fairly standard arguments yield that for periodic boundary motion the energy is uniformly bounded in time.
- Question: Does the energy oscillate, go to an asymptote, stay constant, etc. ?

Speculations on applications to more general Euler flows

- Main idea: Use points in the fluid as “virtual stirrers” or “ghost rods” (Bowen 1978, Thiffeault and Finn 2006).
- Extend TN-theory to get exponential topological growth from a set of points for non-periodic fluid motions (probably doable).
- Show that for typical initial vorticity, a two-dimensional Euler fluid motion always has such orbits (??).
- Provides a new perspective on a version of the Yudovich Hypothesis/Conjecture: For generic initial vorticity a two-dimensional Euler fluid motion satisfies

$$\begin{aligned} \|\Delta \mathbf{u}(\mathbf{x}, t)\|_\infty = \|\nabla \omega_t\|_\infty &\to \infty \\ \|\nabla \omega_t\|_{L^1} &\to \infty \end{aligned}$$

exponentially fast.

Part 4

Optimizing a topological measure of mixing efficiency

with Jason Harrington

Mixing

- Mixing refers to the process by which different materials are intermingled by stirring a fluid. Examples include plastics, cosmetics, rubber, candy, paint, cream in coffee, ...
- In the most basic models one usually considers just the fluid evolution with no diffusion or chemical reactions. This is sometimes called stirring to distinguish it from more detailed models.
- Turbulent fluids mix well, but applications demand using minimal energy and avoiding tearing, bubbles, etc.

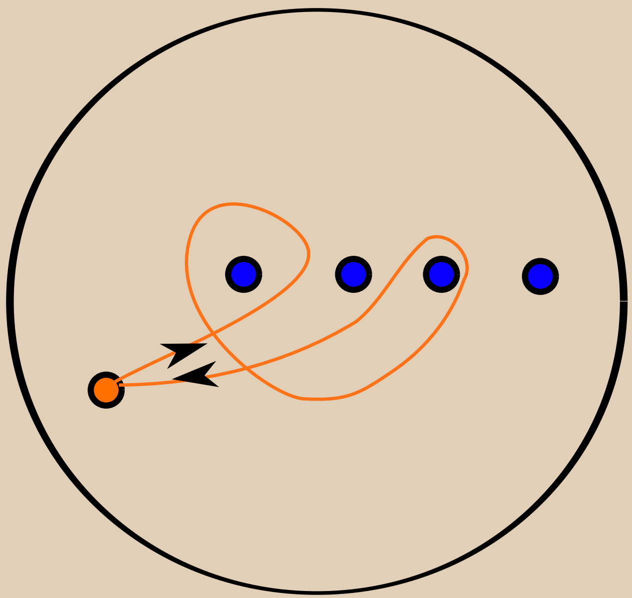
Mixing

- It is clear that stretching (and thus folding) of material lines is necessary for good mixing (but maybe not sufficient).
- We have seen that exponential growth of material lines causes exponential growth of gradients of transported scalars.
- Also, in two dimensions interfaces between materials are one-dimensional, this gives rise to the possibility of enhanced diffusion across the interface.
- Thus it is reasonable to use the global rates of stretching of material lines as an approximate measure of good mixing.
- As we have seen, a pA stirring protocol always causes exponential stretching of material lines with a specified rate of λ^t so we use λ as one measure of mixing.

Entropy efficiency

- Thus we seek protocols which maximize the λ while minimizing the amount of stirrer motion.
- We measure the amount of stirrer motion topologically, but other measures might be more realistic.
- The entropy efficiency is thus the stretch factor λ normalized by taking the k^{th} root where the protocol uses k separate stirrer motions (the unit stirrer motion will be defined shortly).
- Note on terminology: It is common take a logarithm and the entropy of the protocol is defined to be $\log(\lambda)$ and then the efficiency would be $\log(\lambda)/k$.
- We are thus faced with nonlinear, topological optimization problem of maximizing the entropy efficiency among some class of protocols.

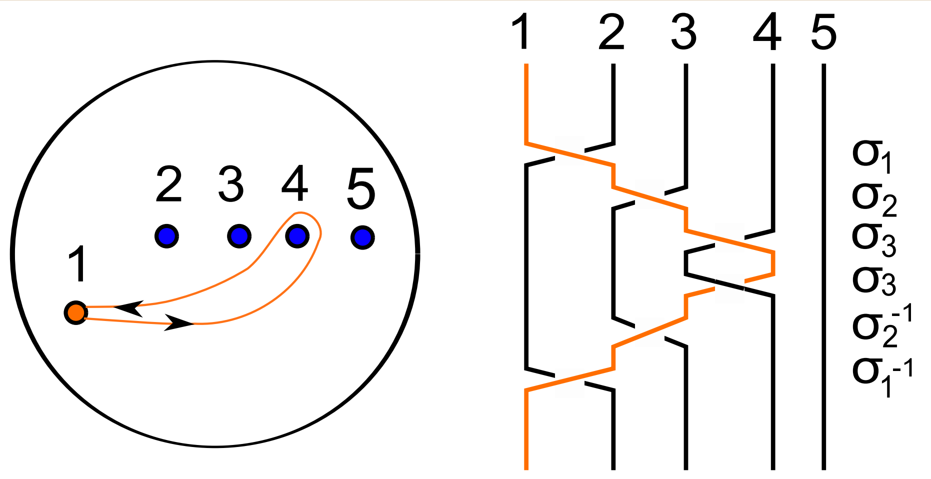
π_i-stirring protocols



We restrict to a special class of stirring protocols. in which a single stirrer S moves around N fixed obstacles.

- Each such protocol is uniquely described by a closed path starting and ending at S.
- Thus the collection of such protocols is naturally isomorphic to $\pi_1(\text{disk minus } N \text{ points})$, the free group on N letters (more on the fundamental group later).

Entropy efficiency of a π_1-protocol



The generator α_4.

We may write each π_1-stirring protocol uniquely as $\eta = \alpha_{i_1}^{\epsilon_{i_1}} \ldots \alpha_{i_k}^{\epsilon_{i_1}}$ where α_j is going around the j^{th} hole once clockwise and $\epsilon_{i_j} = \pm 1$.

- Using $\lambda(\eta)$ to denote the topological stretch rate associated with the protocol η its entropy efficiency is

$$\mathrm{eff}(\eta) = \lambda(\eta)^{\frac{1}{\#(\eta)}},$$

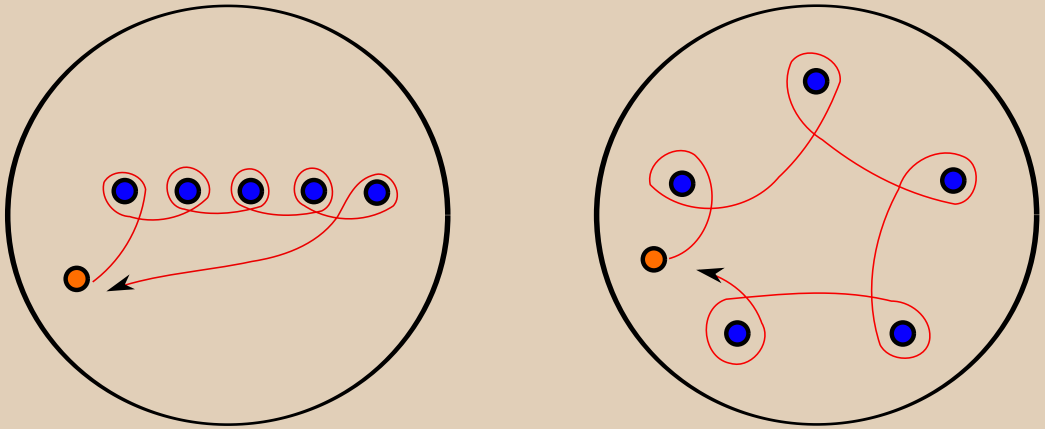
where $\#(\eta)$ is the number of α_j used in η.

Maximum entropy efficiency

- For each N, let $PP(N)$ be the group of all π_1-protocols with N fixed obstacles.
- In this context the maximal entropy efficiency for a given N is

$$\mathrm{Eff}(N) := \sup\{\mathrm{eff}(\eta) : \eta \in PP(N)\}$$

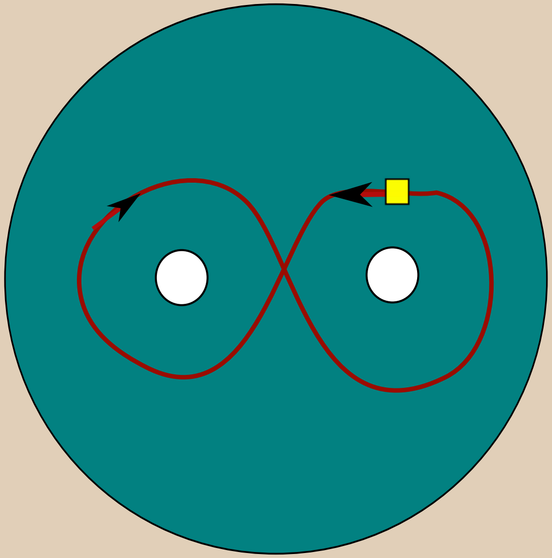
- Numerically, $\mathrm{Eff}(N)$ appears to be achieved by $\mathrm{HSP}_N := \alpha_1\alpha_2\ldots\alpha_N$. What can you prove?



Maximum efficiency; $N = 2$

- The case of two obstacles uses special methods.
- In this case a standard trick (hyperelliptic involution) lifts the problem to linear automorphisms of the two torus.
- Then solve the optimization problem there by hand: the maximal entropy efficiency for $N = 2$ is realized by the protocol $\alpha_1 \alpha_2^{-1}$ and has value $\mathrm{Eff}(2) = 1 + \sqrt{2}$.

The path of the stirrer is a figure eight with an obstacle inside each loop and this protocol is often called the "taffy puller"



Theorem on maximum efficiency when $N > 2$

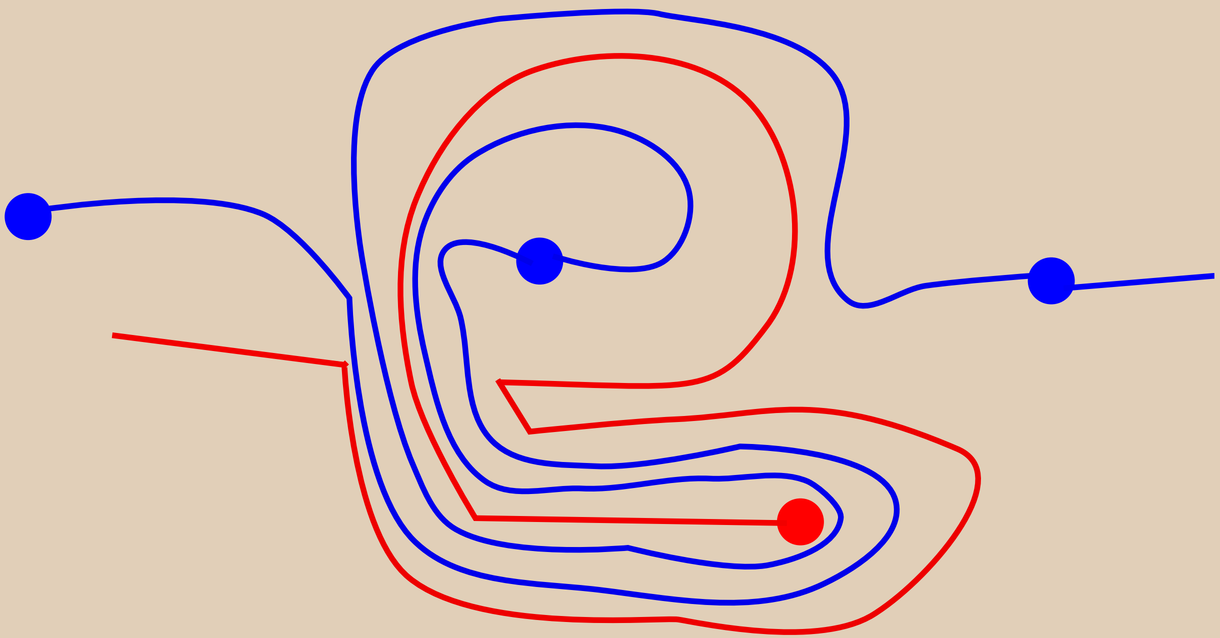
- Roughly, $\mathrm{Eff}(N)$ is asymptotically 3.
- Theorem: There are explicitly defined matrices $H^{(N)}$ and $\hat{H}^{(N)}$ with

$$\begin{aligned}\left(\frac{3^N - 3N - 1}{N}\right)^{1/N} &\leq \rho(H^{(N)})^{1/N} \\ &\leq \mathrm{Eff}(N) \\ &\leq \rho(\hat{H}^{(N)})^{1/N} \leq (3^N - 2)^{1/N},\end{aligned}$$

  where $\rho(M)$ is the spectral radius of a matrix M.
- Thus $\mathrm{Eff}(N) \to 3$ as $N \to \infty$.
- Heuristically, the best you can do is to triple lengths with each stirrer loop.

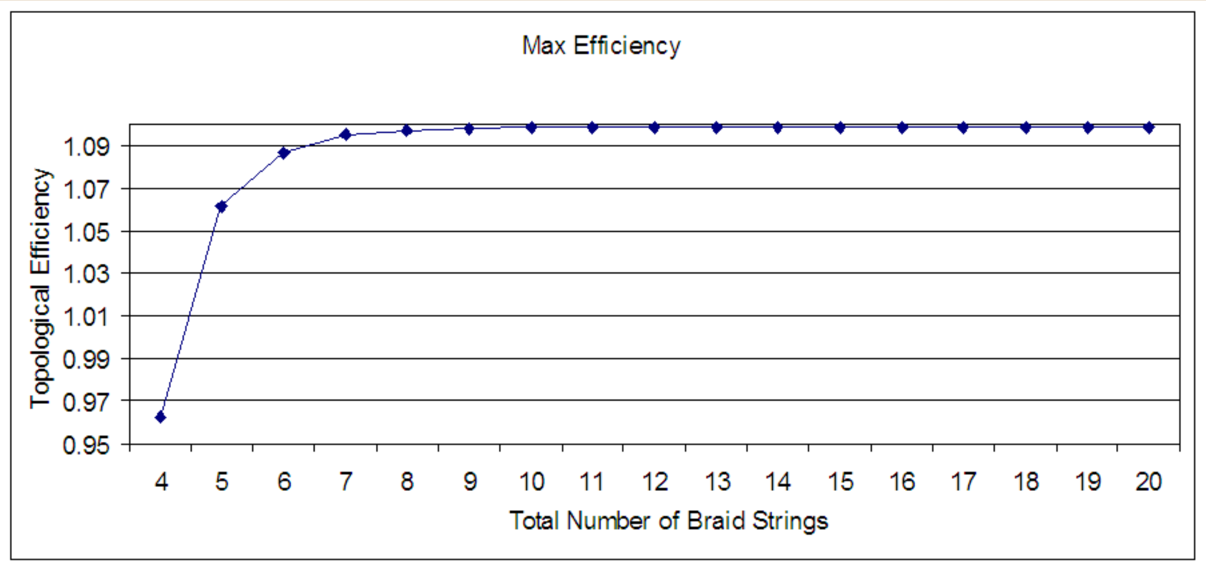
Theorem on maximum efficiency

Intuition: Each looping around a hole adds previous times three, yielding $1 + 3 + \cdots + 3^{N-1} = (3^N - 1)/2$ for N loops.



Remarks on theorem

- Numerical observation: $\rho(H^{(N)}) = 3^N - L(N)$ and $\rho(\hat{H}^{(N)}) = 3^N - \hat{L}(N)$ to high accuracy for linear functions L and $\hat{L}$.
- $\rho(H_N)$ are all Salem numbers and $\rho(\hat{H}_N)$ are all Pisot numbers



Plot of N vs $\log(\mathrm{Eff}(N))$.

Pisot and Salem numbers

- A Pisot number is a real algebraic integer $\alpha > 1$ such that all its Galois conjugates are less than 1 in modulus.
- A Salem number is a real algebraic integer $\alpha > 1$ such that all its Galois conjugates are less than or equal to 1 in modulus and at least one conjugate is on the unit circle. This implies that $1/a$ is a Galois conjugate and all other conjugates are on the unit circle.

Steps in proof of theorem

The lower bound

- Since the entropy efficiency is a max over protocols, any protocol can be used as a lower bound.
- Compute the entropy efficiency of the numerical “winner” HSP_N.
- This computation also involves a linearization, but this time using homology in a special covering space.
- Finally, estimate the spectral radius of a matrix $\hat{H}^{(N)}$.

Steps in proof of theorem

The upper bound

- Transform the topological optimization problem to a nonlinear algebraic one using algebraic topology, specifically, the fundamental group.
- Show that the solution to this problem is bounded above by the solution to its linear analog (the joint spectral radius)
- Prove the needed joint spectral radius is achieved by the matrix $H^{(N)}$.

Entropy efficiency using neighbor swaps

- The natural first question is to consider the maximal stretch rate per unit swap of adjacent stirrers (these are the usual generators of the braid group).
- Finn and Thiffeault (2010) using the argument developed here show that the maximal entropy efficiency with these generators is bounded above by $(1+\sqrt{5})/2$.
- The bound is achieved for 3 rods and for $n > 3$ rods the maximal entropy efficiency decreases for increasing n.
- They also consider a class of protocols where a whole collection of rods move at once.
- We consider here protocols where a single rod moves in which case the maximal entropy efficiency increases for increasing n.

Conclusions

- In two dimensions the proper braiding or knotting of fluid trajectories gives rise to the exponential stretching of topologically essential material lines.
- This, in turn, implies the exponential growth of the maximum deformation and thus of the gradients of any transported scalar.
- Applications to Euler fluid motions then follow from the Helmholtz-Kelvin Theorem.
- One may formulate and in some cases solve the topological optimization problem of maximizing the stretch while minimizing the stirrer motion.