Computing Alexander polynomials using monodromy

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1 Introduction

A general method for computing the Alexander invariants for a finite CW-complex is to start with a finite presentation of the fundamental group of X and to construct a presentation matrix of the Alexander module A_X using Fox Calculus. Although the matrix depends on the presentation of $\pi_1(X)$, the module and its fitting ideals are independent of these choices. In this note, we show how to construct a presentation matrix for the Alexander module of M in the case when M is a 3-manifold fibered over a circle.

2 Background and Definitions

Let X be a finite connected CW-complex. Let $F = \mathrm{H}_1(X;\mathrm{Z})/\text{torsion}$. Let $p \in X$ be a basepoint,

$$\widetilde{\rho} : \widetilde{X} \to X$$

the universal abelian covering of X, i.e., the covering of X defined by the natural epimorphism $\rho : \pi_1(X) \to F$. The *Alexander module* A_X of X is

$$A_X = H_1(\widetilde{X}, \widetilde{\rho}^{-1}(p); \mathrm{Z})$$

considered as a $\mathrm{Z}F$ module.

A presentation matrix for A_X is a matrix

$$M : (\mathrm{Z}F)^s \to (\mathrm{Z}F)^r$$

so that A_X is the quotient of $(\mathrm{Z}F)^r$ by the image of M. The *Alexander ideals* $I_k \subset \mathrm{Z}F$ of A_X are generated by the $(r-k) \times (r-k)$ minors of M.

Theorem 1 *The Alexander ideals I_k do not depend on the presentation matrix M.*

Assume F has rank n. The group ring $\mathrm{Z}F$ is naturally isomorphic to a Laurent polynomial ring $\mathrm{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ and the Alexander ideals define algebraic subsets, $V_k \subset (\mathbb{C}^*)^n$. As we see below, the V_k are useful for computing the first Betti numbers of coverings (or of group cohomology with twisted coefficients). Thus, the V_k have been called *jumping loci.*

When $F = \mathrm{Z}$, then the Alexander polynomial Δ_X is the principle generator of the smallest non-trivial fitting ideal. In general, Δ_X is the generator of the largest principle ideal contained in all the nontrivial fitting ideals. J. Alexander first used Alexander polynomials to distinguish knot complements in 1923. He also showed that they can be computed using Skein relations. Alexander polynomials for knot complements can be computed using Seifert surfaces (1934). The techniques of Skein relations, and Seifert surfaces can be interpreted to include the case when F has higher rank, and there is a distinguished homomorphism $\phi : F \to \mathrm{Z}$. The resulting Alexander polynomial can be thought of as a specialization of the multivariable Alexander polynomial. R. Fox developed his calculus for computing Alexander polynomials from group presentations in 1953. His techniques apply to the case when F has higher rank to compute presentation matrices for the Alexander module with entries in a Laurent polynomial ring isomorphic to $\mathrm{Z}F$.

Since the Alexander module presents the abelianization of the commutator subgroup of the fundamental group of X, we have the following.

Theorem 2 *The Alexander module only depends on the fundamental group $\pi_1(X)$ modulo its second commutator subgroup.*

2.1 Geometric version of Fox Calculus

Start with a finite CW complex X. Construct a cellular chain decomposition

$$\cdots \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0,$$

so that $C_0 = \{p\}$ consists of a single point, C_1 is a bouquet of r oriented circles $y_1, \ldots, y_r$ attached at p, and C_2 is a union of s disks whose boundaries map to C_1. From this information, one has a finite presentation of $G = \pi_1(X)$

$$\mathcal{F}_s \to \mathcal{F}_r \xrightarrow{\alpha} G,$$

where $\mathcal{F}_s$ and $\mathcal{F}_r$ are free groups on s and r generators. Here we interpret the generators of $\mathcal{F}_s$ as the generating 2-cells in C_2, and their images in $R_1, \ldots, R_s \in \mathcal{F}_r$ are representation of their boundaries as elements of $\pi_1(C_1) = \mathcal{F}_r$. Thus, the fundamental group of X can be written as

$$G = \langle y_1, \ldots, y_r \; : \; R_1, \ldots, R_s \rangle.$$

Let $\rho : G \to F$ be any epimorphism of groups. Consider the regular covering $\widetilde{X}_\rho$ defined by ρ. Then $\widetilde{X}_\rho$ has a cell decomposition

$$\cdots \xrightarrow{\widetilde{\partial}_{\rho,3}} \widetilde{C}_{\rho,2} \xrightarrow{\widetilde{\partial}_{\rho,2}} C_{\rho,1} \xrightarrow{\widetilde{\partial}_{\rho,1}} \widetilde{C}_{\rho,0}$$

defined using lifting maps. Choose a lift $\widetilde{p} \in \widetilde{X}_\rho$ of p. Then p determines an identification $\widetilde{C}_{\rho,0} = \mathrm{Z}F$. Let $\widetilde{y}_1, \ldots, \widetilde{y}_r$ be the lifts of $y_1, \ldots, y_r$ to $\widetilde{X}_\rho$ at the basepoint $\widetilde{p}$. Then $\widetilde{C}_{\rho,1}$ is freely generated over $\mathrm{Z}F$ with basis $\widetilde{y}_1, \ldots, \widetilde{y}_r$. The map $\widetilde{\partial}_{\rho,1} : \widetilde{C}_{\rho,1} \to \widetilde{C}_{\rho,0}$ is defined by

$$\widetilde{y}_i \mapsto (\rho(y_i) - 1).$$

Before describing the map $\widetilde{\partial}_{\rho,2} : \widetilde{C}_{\rho,2} \to \widetilde{C}_{\rho,1}$, we first consider the case where $G = \mathcal{F}_r$ is the free group generated by $y_1, \ldots, y_r$, and ρ is the identity map. Identify $\widetilde{C}_{\rho,1}$ with $(\mathrm{Z}\mathcal{F}_r)^r$ as above with basis $\widetilde{y}_1, \ldots, \widetilde{y}_r$ as a $\mathrm{Z}\mathcal{F}_r$ module. Define

$$D : \mathcal{F}_r \to (\mathrm{Z}\mathcal{F}_r)^r$$

so that $D(\gamma)$ is the lift of γ.

Theorem 3 *The map D is determined by the following properties*

$$\begin{aligned} D(y_i) &= \widetilde{y}_i \\ D(fg) &= D(f) + \rho(f)D(g). \end{aligned}$$

The map D is called the *Fox Derivative.*

Let $\overline{\alpha} : (\mathrm{Z}\mathcal{F}_r)^r \to (\mathrm{Z}G)^r$ be the map defined by α, and let $\overline{\rho} : (\mathrm{Z}G)^r \to (\mathrm{Z}F)^r$ be the map defined by ρ. Let

$$\widetilde{\partial}_{\rho,2} : \widetilde{C}_\rho \to \widetilde{C}_{\rho,1}$$

be the lift of ∂_2, identifying $\widetilde{C}_{\rho,1}$ with the free module on $\mathrm{Z}F$ with basis $\widetilde{y}_1, \ldots, \widetilde{y}_r$. The lifting map from $\pi_1(X,p)$ to $\widetilde{C}_1$ at $\widetilde{p}$ extends by $\overline{\alpha}$ to a map on $\mathcal{F}_r$ to give

$$D_\rho : \mathcal{F}_r \to (\mathrm{Z}F)^r.$$

Theorem 4 *The lifting map D_ρ can be decomposed as*

$$D_\rho = \overline{\rho} \circ \overline{\alpha}_r \circ D,$$

where D is the Fox derivative.

If $\rho : G \to F$ be the natural map to the abelianization F of G modulo torsion, then D_ρ is the classical Fox derivative. Since F is a finitely generated, torsion free, abelian group, $\mathrm{Z}F$ can be considered as a ring of Laurent polynomials. The ideal $I_\rho \subset \mathrm{Z}F$ of $r-1 \times r-1$ minors of $\widetilde{\partial}_{\rho,2}$ is called the *Alexander ideal*, and the generator of the largest principle ideal contained in I_1 is the classical multivariable Alexander polynomial. Twisted Alexander invariants are defined by letting ρ map G to a subgroup of $\mathrm{GL}_n(\mathbb{R})$.

2.2 Alexander invariants and group cohomology with twisted coefficients.

Alexander invariants can also be defined starting from an abstract finitely presented group. Given a finitely presented group G with r generators $y_1, \ldots, y_r$ and s relations $R_1, \ldots, R_s$, let X be the 2-dimensional CW complex with one 0-cell p, r 1-cells attached to p at their end points, and s 2-cells attached along their boundaries to the 1-cells by a map determined by R_i. The Alexander invariants for G are then defined as above using X.

Alternately, we can make a dual but very similar description for the group cohomology of a finitely presented group. For $\rho \in \widehat{F}$, let $\mathbb{C}_\rho$ be the $\mathrm{Z}G$-module where each $g\ inG$ acts on $\mathbb{C}$ by multiplication by $\rho(g)$. (See [1] Section 2.2 for details.)

One can interpret the V_k in this context as follows.

Theorem 5 *For $\phi \in \widehat{F}$ and ϕ not equal to the identity character, $\phi \in V_k$ if and only if the rank of $\mathrm{H}^1(X, \mathbb{C}_\rho)$ is greater than or equal to k.*

2.3 Alexander invariants and the first Betti number of finite abelian coverings

Let G be any finitely presented group with presentation

$$\mathcal{F}_s \to \mathcal{F}_r \xrightarrow{q} G.$$

Let $\widehat{G} = \mathrm{Hom}(G, \mathbb{C}^*)$, identify $(\mathbb{C}^*)^r$ with $\mathrm{Hom}(\mathcal{F}_r, \mathbb{C}^*)$, and let

$$\widehat{G} \to (\mathbb{C}^*)^r$$

be the natural map defined by composition with q.

For any finite quotient $\alpha : G \to T$ there is a natural inclusion

$$\widehat{\alpha} : \widehat{T} \hookrightarrow \widehat{G} = (\mathbb{C}^*)^r.$$

The Alexander ideals $I_{\rho,k}$ define subvarieties $V_k \subset (\mathbb{C}^*)^r$, called the *Alexander strata.*

Theorem 6 *Let T be a finite abelian group, $\phi : \pi_1(X) \to T$ and epimorphism, and $\rho_\phi : X_\phi \to X$ the corresponding finite abelian covering. Then $H_1(X_\phi, \rho_\phi^{-1}(p); \mathrm{Z})$ has rank given by*

$$b_1(X_\phi, \rho_\phi^{-1}(p)) = b_1(X) + \sum_{k=1}^{r} |V_k \cap \widehat{\alpha}(\widehat{T}) \setminus \widehat{1}|$$

where $\widehat{1} \in (\mathbb{C}^*)^r$ is the identity element.

See [1], Proposition 2.5.6.

Corollary 7 *The first Betti numbers of finite abelian coverings of a finite CW complex X are determined by the Alexander module A_X and the corresponding stratification of $\widehat{G}$.*

2.4 The first Alexander ideal, and the Alexander polynomial

The first Alexander ideal I_1 has a particularly simple form. The following useful lemma is contained in the proof of Theorem 5.1 in [2].

Lemma 8 *Let $F = \mathrm{Z}^n$, and let*

$$(\mathrm{Z}F)^s \xrightarrow{M} (\mathrm{Z}F)^r \xrightarrow{\widetilde{\alpha}} (\mathrm{Z}F)$$

be a sequence of $\mathrm{Z}F$ module homomorphisms such that $\alpha \circ M$ is the zero map. Recall that $\mathrm{Z}F$ is a unique factorization domain. Let $g_1, \ldots, g_r$ be basis elements of $(\mathrm{Z}F)^r$. Assume that $\widetilde{\alpha}(g_i)$ is irreducible for each $i = 1, \ldots, r$. Let m be the ideal in $\mathrm{Z}F$ generated by the image of the free generators of $(\mathrm{Z}F)^r$ under the map $\widetilde{\alpha}$. The ideal of $(r-1) \times (r-1)$ minors of M is of the form $\Delta \cdot m$, for some polynomial $\Delta \in \mathrm{Z}F$.

Proof. We will show that each $(r-1) \times (r-1)$ minor is of the form $\Delta \cdot \alpha(g)$ for some basis element g of $(\mathrm{Z}F)^r$. For the purposes of computing the minor determinants, we may assume that M is an $r \times (r-1)$ matrix by removing some columns, so that

$$M : (\mathrm{Z}F)^{r-1} \to (\mathrm{Z}F)^r.$$

Since $\alpha \circ M = 0$, we know that

$$\sum_{i=1}^{r} \alpha(g_i) a_{i,j} = 0 \tag{1}$$

where $M = [a_{i,j}]$.

For each $i = 1, \ldots, r$, let Δ_i be the determinant of the matrix M with the ith row removed. Let i and i' be the indices of two rows of M. We will show that

$$\alpha(g_i)\Delta_{i'} = \pm\alpha(g_{i'})\Delta_i. \tag{2}$$

Assume that $i \neq i'$. By (1) we have

$$\alpha(g_i)a_{i,j} = -\sum_{k\neq i} \alpha(g_k)a_{i,j}.$$

This means that if we multiply the ith row of $M_{i'}$ by $\alpha(g_i)a_{i,j}$, then the ith row is a linear combination of the other rows of $M_{i'}$ and $\alpha(g_{i'})$ times the $i'$the row of M_i. The equation (2) follows.

Since $\alpha(g_i)$ and $\alpha(g_{i'})$ are irreducible, (2) implies that up to units in ZF

$$\Delta_{i'} = \alpha(g_i)\Delta$$

and

$$\Delta_i = \alpha(g_{i'})\Delta$$

for some $\Delta \in$ ZF that is independent of i and i'.

□

3 Fibered complexes

Let $\phi : X \to S^1$ be a fibration with fiber S, and monodromy $\psi : S \to S$. For simplicity we will assume that $\pi_1(S)$ is free. (The same results hold if $\pi_1(S)$ is the quotient of a free group by the conjugates of an element in its commutator subgroup.) Then the exact sequence

$$\pi_1(S) \to \pi_1(X) \to \pi_1(S^1) = \mathrm{Z}$$

splits, giving the following presentation for $\pi_1(X)$

$$\langle x_1, \ldots, x_n, w \ : \ wx_iw^{-1} = \psi_*(x_i), \quad i = 1, \ldots, n\rangle$$

Then X is isomorphic to a chain complex of the form

$$\cdots \xrightarrow{\partial_3} C_2^n \xrightarrow{\partial_2} C_1^{n+1} \xrightarrow{\partial_1} C_0.$$

where the subscripts indicate the dimension of the chains, and the superscripts indicate the rank of C_k as a free module over Z.

Let $F_\phi = \mathrm{Z}^r$, where r is the rank of $\mathrm{H}^1(S;\mathrm{Z})^{\psi-\mathrm{inv}}$, the elements of $\mathrm{H}^1(S;\mathrm{Z})$ that are invariant under the action of ψ. Then generators for $\mathrm{H}^1(S;\mathrm{Z})^{\psi-\mathrm{inv}}$ define a surjection

$$\rho : \pi_1(S) \to F_\phi.$$

Let $S_\phi \to S$ be the corresponding unbranched covering.

1. $F = \mathrm{H}_1(X; \mathrm{Z})$ decomposes into a direct sum:

$$F = F_\phi \oplus \langle u \rangle$$

where u is the image of w in $\mathrm{H}_1(X; \mathrm{Z})$ under abelianization.

2. The maximal abelian covering $\widetilde{X}$ of X is homeomorphic to $S_\phi \times \mathbb{R}$.

Choose a basepoint $p \in S$, and $\widetilde{p}$ a basepoint in S_ϕ that maps to p under the covering map. Let $\widetilde{x}_1, \ldots, \widetilde{x}_n$ be lifts of $x_1, \ldots, x_n$ to $\widetilde{p}$. Then we can write $\widetilde{X}$ as a chain complex of the form

$$\cdots \xrightarrow{\widetilde{\partial}_3} \widetilde{C}_2 \xrightarrow{\widetilde{\partial}_2} \widetilde{C}_1 \xrightarrow{\widetilde{\partial}_1} \widetilde{C}_0$$

where

$$\begin{aligned} \widetilde{C}_2 &= (\mathrm{Z}F)^n \\ \widetilde{C}_1 &= (\mathrm{Z}F)^{n+1} \end{aligned}$$

and the maps $\widetilde{\partial}_i$ are $\mathrm{Z}F$-module homomorphisms.

The map $\widetilde{\partial}_1 : \widetilde{C}_1 \to \widetilde{C}_0$ is given by

$$\begin{aligned} \widetilde{x}_i &\mapsto (\rho(x_i) - 1)\widetilde{p} \\ \widetilde{w} &\mapsto (\rho(w) - 1)\widetilde{p} \end{aligned}$$

So in matrix form $\widetilde{\partial}_1$ is given by

$$\widetilde{\partial}_1 = [\rho(w) - 1, \rho(x_1) - 1 \cdots \rho(x_n) - 1].$$

The image of $\widetilde{\partial}_1$ is the augmentation of ideal of $\mathrm{Z}F$.

We now turn to $\widetilde{\partial}_2$. Consider any lift of $\psi : \pi_1(X, p) \to \pi_1(X, p)$ to an automorphism of free groups

$$\Psi : \langle w, x_1, \ldots, x_n \rangle \to \langle w, x_1, \ldots, x_n \rangle.$$

Then

$$\begin{aligned} \widetilde{\partial}_2 &= \left[D_\rho(w x_i w^{-1} (\Psi(x_i)^{-1}) \right] \\ &= \begin{bmatrix} \rho(x_1) - 1 & \rho(x_2) - 1 & \cdots & \rho(x_n) - 1 \\ u - D_{\rho,1}(\Psi(x_1)) & -D_{\rho,1}(\Psi(x_2)) & \cdots & -D_{\rho,1}(\Psi(x_n)) \\ -D_{\rho,2}(\Psi(x_1) & u - D_{\rho,2}(\Psi(x_2) & \cdots & -D_{\rho,2}(\Psi(x_n) \\ \cdots & \cdots & \cdots & \cdots \\ -D_{\rho,n}(\Psi(x_1) & -D_{\rho,n}(\Psi(x_2) & \cdots & u - D_{\rho,n}(\Psi(x_n) \end{bmatrix}. \end{aligned}$$

Lemma 9 *The $n \times n$ minors of $\widetilde{\partial}_2$ are of the form*

$$(u - 1)\Delta(u, t_1, \ldots, t_n)$$

and

$$(t_i - 1)\Delta(u, t_1, \ldots, t_n), \quad \textit{for } i = 1, \ldots, n,$$

where $\Delta(u, t_1, \ldots, t_n)$ is some fixed polynomial in $\mathrm{Z}[u^{\pm 1}, t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ that only depends up to units on X.

Corollary 10 *The polynomial Δ is the multivariable Alexander polyomial of X.*

4 Example

Let L be the closure of the $\sigma_1\sigma_2^{-1}$ braid together with an encircling link. Figure 1 shows two diagrams for the link L, showing that the link is symmetric in its components. Let $M = S^3 \setminus N(L)$, where $N(L)$ is a union of tubular neighborhoods of the components of L. (The link L is given in Rolfsen's knot tables as the 6_2^2 link.)

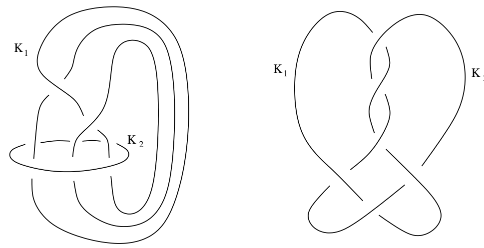

Figure 1: The 6_2^2 link.

By construction M_L is fibered over the circle with fiber a disk S with three punctures, and the monodromy is the braid monodromy defined by $\phi = \sigma_1\sigma_2^{-1}$. Choose a basepoint p on S and let x, y, z be paths on S with initial and endpoints at p passing along a path from p to a point near a puncture, encircling the puncture counterclockwise, and returning along the same path back to p. The result of the action of the monodromy of ϕ on x, y, z is shown in Figure 2.

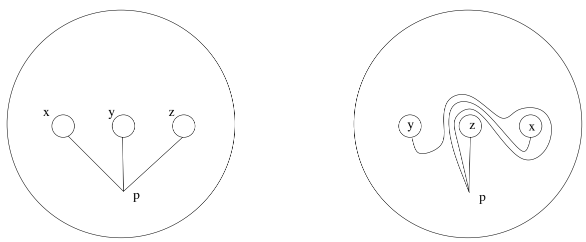

Figure 2: The braid monodromy action on $\pi_1(S)$.

Let $t, u \in \mathrm{H}_1(M_L; \mathrm{Z})$ be oriented meridian loops around the components of L, with t a meridian for the braid closure K_1, and u a meridian for the encircling link K_2. Then $\mathrm{H}_1(M; \mathrm{Z})$ is the free abelian group generated by t and u. Under the abelianization map,

$$\pi_1(M_L) \to \mathrm{H}_1(M_L; \mathrm{Z})$$

the loops x, y, z each map to t. Let $w \in \pi_1(M_L)$ be a lift of u under the abelianization map. Then $\pi_1(M_L)$ is generated by x, y, z, w and has relations

$$\begin{aligned} R_1 : wxw^{-1} &= y^{-1}zy \\ R_2 : wyw^{-1} &= y^{-1}zyx^{-1}y^{-1}z^{-1}y \\ R_3 : wzw^{-1} &= y. \end{aligned}$$

The Fox derivatives for the Alexander module A_L of M_L is given by

$$\begin{aligned} \mathrm{D}R_1 &= (1-t)\mathrm{d}u + u\mathrm{d}x + (t^{-1}-1)\mathrm{d}y - t^{-1}\mathrm{d}z \\ \mathrm{D}R_2 &= (1-t)\mathrm{d}u - t\mathrm{d}x + (u+t+t^{-1}-2)\mathrm{d}y + (1-t^{-1})\mathrm{d}z \\ \mathrm{D}R_3 &= (1-t)\mathrm{d}u - \mathrm{d}y + u\mathrm{d}z. \end{aligned}$$

The Alexander polynomial of the Alexander module A_L is the greatest common divisor of the 3×3 minors of the Fox matrix

$$\begin{bmatrix} 1-t & 1-t & 1-t \\ u & -t & 0 \\ t^{-1}-1 & u+t+t^{-1}-2 & -1 \\ -t^{-1} & 1-t^{-1} & u \end{bmatrix}.$$

A calculation shows that the minors obtained by deleting the first row gives

$$(u-1)(1-u(1-t-t^{-1})+u^2)$$

while the others are given by

$$\pm(t-1)(1-u(1-t-t^{-1})+u^2).$$

References

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[2] C. McMullen. The Alexander polynomial of a 3-manifold and the thurston norm on cohomology. *Ann. Scient. Éc. Norm. Sup.*, 35:153–171, 2002.