A DISCONNECTED DEFORMATION SPACE OF RATIONAL MAPS

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ABSTRACT. Let $f : (S^2, A) \to (S^2, B)$ be an orientation-preserving branched covering of pairs where A and B are finite sets containing at least 3 points, B contains the critical values of f, and $A \subseteq B$. We study the *deformation space* $\mathcal{D}_f$ consisting of classes of marked rational maps $F : (\mathbb{P}^1, A') \to (\mathbb{P}^1, B')$ that are combinatorially equivalent to f. In the case $A = B$, Thurston gave a topological criterion for which $\mathcal{D}_f \neq \emptyset$, and he proved that $\mathcal{D}_f$ is always connected. We show that if $A \subsetneq B$, then $\mathcal{D}_f$ need not be connected. We exhibit a family of quadratic rational maps for which the associated deformation spaces are disconnected; in fact they have infinitely many components. In particular, we construct explicit automorphisms of $\mathcal{D}_f$ that cyclically permute an infinite set of connected components.

1. Introduction

Let $f : (S^2, A) \to (S^2, B)$ be an orientation-preserving branched covering of pairs where A and B are finite sets containing at least 3 points, and B contains the critical values of f. In this paper, we study spaces of rational maps equivalent to f from two different perspectives. The first perspective is 'nondynamical': we let $\mathcal{U}_f$ be the space of marked rational maps that are Hurwitz equivalent to f, where A and B are not related. The second perspective is 'dynamical', where we identify domain and range and assume $A \subseteq B$. This determines a subspace $\mathcal{D}_f \subseteq \mathcal{U}_f$ of rational maps combinatorially equivalent to f called the *deformation space* of f. In the purely dynamical setting where $A = B$, W. Thurston gave a criterion for $\mathcal{D}_f$ to be nonempty and established the following result.

Theorem 1.1 (W. Thurston [DH], [BCT])**.** *If $A = B$, then $\mathcal{D}_f$ is connected, and if $\mathcal{D}_f$ is nonempty and does not contain an element of Lattès type*[1]*, then $\mathcal{D}_f$ is a single point.*

More generally, when $A \subseteq B$, Epstein proved the following theorem.

Theorem 1.2 (A. Epstein [E])**.** *If $\mathcal{D}_f$ is nonempty, and does not contain an element of Lattès type, then $\mathcal{D}_f$ is a smooth complex submanifold of $\mathcal{U}_f$ of dimension $|B - A|$.*

The main result in this paper is that $\mathcal{D}_f$ need not be connected in general. Assume $\mathcal{D}_f$ is nonempty, and let $u_0 \in \mathcal{D}_f$. The covering map $\omega : (\mathcal{U}_f, u_0) \to (\mathcal{W}_f, v_0)$ defined by forgetting the markings restricts to a covering map $\nu : (\mathcal{D}_f, u_0) \to (\mathcal{V}_f, v_0)$, and we have a commutative

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[1]See [M1] for the definition of a Lattès map.

diagram:

$$\begin{tikzcd} (\mathcal{D}_f, u_0) \arrow[r, hook] \arrow[d, "\nu"'] & (\mathcal{U}_f, u_0) \arrow[d, "\omega"] \\ (\mathcal{V}_f, v_0) \arrow[r, hook] & (\mathcal{W}_f, v_0) \end{tikzcd}$$

We study the connected components of $\mathcal{D}_f$ using the covering map ν. Let S_f be the group of covering automorphisms of ν and let E_{u_0} be the setwise stabilizer of the connected component of $\mathcal{D}_f$ containing u_0. We characterize S_f as a particular subgroup of the pure mapping class group of (S^2, B), and we show that S_f acts transitively on fibers of ν. It follows that there is a bijection between the connected components of $\mathcal{D}_f$ and the set of cosets $\mathrm{S}_f/\mathrm{E}_{u_0}$.

We prove that $\mathcal{D}_f$ is disconnected when f belongs to a particular subspace of the space of quadratic rational maps. Let $\mathrm{M}_2^{\mathrm{cm}}$ be the moduli space of critically-marked quadratic rational maps. Let $\mathrm{Per}_4(0) \subseteq \mathrm{M}_2^{\mathrm{cm}}$ be the subspace of maps with a marked superattracting 4-cycle (see [M2]), and define $\mathrm{Per}_4(0)^* \subseteq \mathrm{Per}_4(0)$ to be the subspace for which the superattracting 4-cycle contains only one critical point. Let f represent an element of $\mathrm{Per}_4(0)^*$, let A be the set of points in the marked superattracting 4-cycle, and let $B := A \cup \{b\}$ where $b \notin A$ is the other critical value of f. Then $f : (\mathbb{P}^1, A) \to (\mathbb{P}^1, B)$ defines a branched covering of pairs and $\mathcal{D}_f$ has a canonical basepoint.

Theorem 1.3. *If $\langle f \rangle \in \mathrm{Per}_4(0)^*$, then $\mathcal{D}_f$ has infinitely many connected components.*

Remark 1.4. Firsova, Kahn, and Selinger have given a different and independent proof that $\mathcal{D}_f$ is disconnected for $\langle f \rangle \in \mathrm{Per}_4(0)^*$. Rees also has a substantial body of work related to the topology of deformation spaces [R].

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2. Spaces of rational maps and their modular groups

In this section, we begin with a branched covering of pairs $f : (S^2, A) \to (S^2, B)$. Here we assume A and B are finite, each containing at least 3 points, and B contains the critical values of f. We first define the spaces of rational maps $\mathcal{U}_f$ and $\mathcal{W}_f$, and the regular covering map $\omega : \mathcal{W}_f \to \mathcal{U}_f$. We then suppose $A \subseteq B$, define the deformation space $\mathcal{D}_f$, and study the regular covering map $\nu : \mathcal{D}_f \to \mathcal{V}_f$ induced by ω. The language of covering maps allows us to translate the problem of comparing subgroups of covering automorphisms to subgroups of the fundamental groups of spaces. We then describe how equalizers play an important role in this discussion.

2.1. Rational maps marked by a branched covering.

Let $f : (S^2, A) \to (S^2, B)$ be a branched covering of pairs. A rational map $F : (\mathbb{P}^1, A') \to (\mathbb{P}^1, B')$ is *Hurwitz equivalent* to f if there is a commutative diagram

$$\begin{tikzcd} (S^2, A) \arrow[r, "\psi"] \arrow[d, "f"'] & (\mathbb{P}^1, A') \arrow[d, "F"] \\ (S^2, B) \arrow[r, "\phi"] & (\mathbb{P}^1, B') \end{tikzcd}$$

where ϕ and ψ are orientation-preserving homeomorphisms of pairs. The commutative diagram is called an *f-marking* of F and is denoted (ψ, ϕ, F). The set of equivalence classes $[\psi, \phi, F]$ of f-markings (ψ, ϕ, F) forms a space we denote by $\mathcal{U}_f$. Here two f-markings (ψ_1, ϕ_1, F_1) and (ψ_2, ϕ_2, F_2) are *equivalent* if there is a commutative diagram

$$\begin{tikzcd}
(S^2, A) \arrow[rrr, "\psi_2"] \arrow[ddd, "f"'] \arrow[dr, "e'"] & & & (\mathbb{P}^1, A'_2) \arrow[ddd, "F_2"] \arrow[dl, "\alpha"'] \\
& (S^2, A) \arrow[r, "\psi_1"] \arrow[d, "f"'] & (\mathbb{P}^1, A'_1) \arrow[d, "F_1"] & \\
& (S^2, B) \arrow[r, "\phi_1"] & (\mathbb{P}^1, B'_1) & \\
(S^2, B) \arrow[rrr, "\phi_2"] \arrow[ur, "e"] & & & (\mathbb{P}^1, B'_2) \arrow[ul, "\beta"']
\end{tikzcd}$$

where e' is isotopic to the identity rel A, e is isotopic to the identity rel B, and α and β are Möbius transformations.

The space $\mathcal{U}_f$ may be thought of as the *Teichmüller space* of f-marked rational maps. In fact, it can be canonically identified with the Teichmüller space $\mathcal{T}_B$ of (S^2, B) which we now define.

2.2. **Teichmüller and moduli spaces.** Recall that the *Teichmüller space* of (S^2, A), denoted $\mathcal{T}_A$, is the space of orientation-preserving homeomorphisms or *markings*

$$\phi : (S^2, A) \to (\mathbb{P}^1, A')$$

up to pre-composition by isotopy equivalence rel A and post-composition by Möbius transformations.

The branched covering $f : (S^2, A) \to (S^2, B)$ defines a pullback map $\sigma_f : \mathcal{T}_B \to \mathcal{T}_A$ (see [DH]).

Proposition 2.1. *The space $\mathcal{U}_f$ is naturally isomorphic to the graph of σ_f.*

The Teichmüller space $\mathcal{T}_A$ has a natural quotient space $\mathcal{M}_A$, the *moduli space* of (S^2, A); that is, the set of all injective maps $i : A \to \mathbb{P}^1$ up to post-composition by Möbius transformations. The map $\mathcal{T}_A \to \mathcal{M}_A$ which sends the equivalence class of ϕ to the equivalence class of $\phi|_A$ is a universal covering map. The *pure mapping class group* of (S^2, A) is the group of isotopy classes rel A of orientation-preserving homeomorphisms $(S^2, A) \to (S^2, A)$ that fix A pointwise. This group acts freely and properly discontinuously on $\mathcal{T}_A$ by pre-composition and is isomorphic to the *modular group* Mod_A, the group of covering automorphisms of $\mathcal{T}_A \to \mathcal{M}_A$.

2.3. **Moduli space of rational maps Hurwitz equivalent to f.** We define a moduli space $\mathcal{W}_f$, a natural quotient map $\omega : \mathcal{U}_f \to \mathcal{W}_f$, and an associated modular group, which we will denote by L_f.

Consider the space of triples (i, j, F) where $i : A \to \mathbb{P}^1$ and $j : B \to \mathbb{P}^1$ are injective maps, and $F : (\mathbb{P}^1, i(A)) \to (\mathbb{P}^1, j(B))$ is a rational map. Two such triples (i_1, j_1, F_1) and (i_2, j_2, F_2)

are *equivalent* if there is a commutative diagram

$$\begin{tikzcd}
A \arrow[r, "i_1"] \arrow[rr, bend left, "i_2"] \arrow[d, "f|_A"] & \mathbb{P}^1 \arrow[r, "\alpha"] \arrow[d, "F_1"] & \mathbb{P}^1 \arrow[d, "F_2"] \\
B \arrow[r, "j_1"] \arrow[rr, bend right, "j_2"'] & \mathbb{P}^1 \arrow[r, "\beta"] & \mathbb{P}^1
\end{tikzcd}$$

where α and β are Möbius transformations. Let $[i, j, F]$ denote the equivalence class of (i, j, F). Consider the map which sends $[\psi, \phi, F] \in \mathcal{U}_f$ to the equivalence class $[\psi|_A, \phi|_B, F]$. Let $\mathcal{W}_f$ be the image of $\mathcal{U}_f$, and define ω to be

$$\begin{array}{rcl}
\omega : \mathcal{U}_f & \to & \mathcal{W}_f \\
{[\psi, \phi, F]} & \mapsto & [\psi|_A, \phi|_B, F].
\end{array}$$

The space $\mathcal{W}_f$ can be thought of as the *moduli space* of rational maps which are Hurwitz equivalent to f. The map $\omega : \mathcal{U}_f \to \mathcal{W}_f$ is a regular covering map whose group of covering automorphisms is the group L_f, which we define below.

2.4. **The liftables.** Let $h : (S^2, B) \to (S^2, B)$ be an orientation-preserving homeomorphism that fixes B pointwise. We say that h is *liftable* if there is an orientation-preserving homeomorphism $h' : (S^2, A) \to (S^2, A)$ fixing A pointwise so that the following diagram commutes.

$$\begin{tikzcd}
(S^2, A) \arrow[r, "h'"] \arrow[d, "f"] & (S^2, A) \arrow[d, "f"] \\
(S^2, B) \arrow[r, "h"] & (S^2, B).
\end{tikzcd}$$

Any lift h' is unique up to covering automorphisms. The condition that h' fixes A makes h' unique (cf. [K]). By the homotopy-lifting property, the group of liftable homeomorphisms descends to the corresponding group of *liftables* associated to f

$$\mathrm{L}_f := \{[h] \in \mathrm{Mod}_B \mid h \text{ is liftable}\}.$$

This defines the *lifting homomorphism*

$$\begin{array}{rcl}
\Phi_f : \mathrm{L}_f & \to & \mathrm{Mod}_A \\
{[h]} & \mapsto & [h'].
\end{array}$$

There is a natural action of L_f on $\mathcal{U}_f$ given by

$$[h] \cdot [\psi, \phi, F] \mapsto [\psi \circ h', \phi \circ h, F].$$

See [K] for a proof of the following proposition.

Proposition 2.2. *The map $\omega : \mathcal{U}_f \to \mathcal{W}_f$ is a regular covering map with group of covering automorphisms isomorphic to L_f.*

2.5. **Equalizers.** Much of the discussion in this paper will use the language of equalizers; we now recall the definition. Let $f, g : X \to Y$ be two maps between sets X and Y. The *equalizer* of f and g is

$$\mathcal{E}\mathrm{q}(f, g) := \{x \in X \mid f(x) = g(x)\}.$$

Note that the set $\mathcal{E}\mathrm{q}(f, g)$ may be empty.

2.6. **The deformation space.** Assume that $A \subseteq B$. There are two natural maps

$$\tau_1, \tau_2 : \mathcal{U}_f \to \mathcal{T}_A$$

given by

$$\tau_1 : [\psi, \phi, F] \mapsto [\psi] \quad \text{and} \quad \tau_2 : [\psi, \phi, F] \mapsto [\phi]_A$$

where $[\phi]_A$ denotes the equivalence class of ϕ in $\mathcal{T}_A$.

An f-marking (ψ, ϕ, F) is a *combinatorial equivalence* if $[\phi]_A = [\psi]$ and $\psi|_A = \phi|_A$. Then $[\psi, \phi, F]$ satisfies $\tau_1([\psi, \phi, F]) = \tau_2([\psi, \phi, F])$ if and only if $[\psi, \phi, F]$ contains a combinatorial equivalence. We define the *deformation space* $\mathcal{D}_f$ to be

$$\mathcal{D}_f := \mathcal{E}\mathrm{q}(\tau_1, \tau_2).$$

We can think of $\mathcal{D}_f$ as the *dynamical Teichmüller space* defined by f.

2.7. **Moduli space of rational maps combinatorially equivalent to f.** We denote the image of $\mathcal{D}_f$ in $\mathcal{W}_f$ as $\mathcal{V}_f := \omega(\mathcal{D}_f)$. The space $\mathcal{V}_f$ can be thought of as the the *dynamical moduli space* defined by f, or as the *moduli space* of rational maps which are combinatorially equivalent to f. We define the associated modular group in the next section.

There are two natural maps

$$\rho_1, \rho_2 : \mathcal{W}_f \to \mathcal{M}_A$$

given by

$$\rho_1 : [\psi|_A, \phi|_B, F] \mapsto [\psi|_A] \quad \text{and} \quad \rho_2 : [\psi|_A, \phi|_B, F] \mapsto [\phi|_A].$$

We have

$$\mathcal{V}_f \subseteq \mathcal{E}\mathrm{q}(\rho_1, \rho_2),$$

and $\mathcal{V}_f$ is a union of connected components of $\mathcal{E}\mathrm{q}(\rho_1, \rho_2)$.

2.8. **The special liftables.** The assumption $A \subseteq B$ determines a subgroup of L_f which preserves $\mathcal{D}_f$. Given $g \in \mathrm{Mod}_B$, we denote by g_A the mapping class in Mod_A defined by g. We define the *special liftables* to be

$$\mathrm{S}_f := \{\ell \in \mathrm{L}_f \mid \Phi_f(\ell) = \ell_A\}.$$

Suppose $\mathcal{D}_f \neq \emptyset$, and consider the map $\nu : \mathcal{D}_f \to \mathcal{V}_f$ given by restricting $\omega : \mathcal{U}_f \to \mathcal{W}_f$. As established in Proposition 2.5, ν is a regular covering map whose group of covering automorphisms is isomorphic to S_f. We can therefore think of S_f as the *dynamical modular group* of f.

Lemma 2.3. *The group S_f preserves $\mathcal{D}_f$.*

Proof. An f-marking (ψ, ϕ, F) represents an element in $\mathcal{D}_f$ if and only if $[\psi] = [\phi]_A$ in $\mathcal{T}_A$. Because $[h] \in \mathrm{S}_f$, $[\phi \circ h]_A = [h]_A \cdot [\phi]_A = [h'] \cdot [\psi] = [\psi \circ h']$, so $[h] \cdot [\phi, \psi, F] \in \mathcal{D}_f$. □

Lemma 2.4. *For $[h] \in \mathrm{L}_f$, if $[h] \cdot \mathcal{D}_f \cap \mathcal{D}_f \neq \emptyset$, then $[h] \in \mathrm{S}_f$.*

Proof. Take $[h] \in \mathrm{L}_f$, and let $[\psi, \phi, F] \in \mathcal{D}_f$. Assume that $[h] \cdot [\psi, \phi, F] \in \mathcal{D}_f$. Then we have

$$[\psi \circ h'] = [\phi \circ h]_A \quad \text{equivalently} \quad [h'] \cdot [\psi] = [h]_A \cdot [\phi]_A$$

in $\mathcal{T}_A$. Because $[\psi] = [\phi]_A$ and because the action of Mod_A on $\mathcal{T}_A$ is free, we have $[h'] = [h_A]$, so $[h] \in \mathrm{S}_f$. □

Proposition 2.5. *The map $\nu : \mathcal{D}_f \to \mathcal{V}_f$ is a regular covering map with covering automorphisms S_f.*

Proof. Lemmas 2.3 and 2.4 imply that $\mathcal{V}_f$ is the quotient of $\mathcal{D}_f$ by the action of S_f. □

2.9. **Connected components of $\mathcal{D}_f$.** Assume $\mathcal{D}_f \neq \emptyset$, and let $u_0 \in \mathcal{D}_f$ be a basepoint. Let $v_0 := \nu(u_0)$, let $\mathcal{V}_0$ be the connected component of $\mathcal{V}_f$ containing v_0, let $\mathcal{D}_0 := \nu^{-1}(\mathcal{V}_0)$ and let $\nu_0 := \nu|_{\mathcal{D}_0}$. Proposition 2.5 implies that

$$\nu_0 : (\mathcal{D}_0, u_0) \to (\mathcal{V}_0, v_0)$$

is a regular covering map with covering automorphisms S_f. Consider the defining map for the covering ν_0

$$\mathfrak{h}_0 : \pi_1(\mathcal{V}_0, v_0) \to \mathrm{S}_f$$

defined by path-lifting. The image E_{u_0} of $\mathfrak{h}_0$ is the group of covering automorphisms of ν_0 restricted to the connected component $\mathcal{D}_{u_0}$ of $\mathcal{D}_0$ containing u_0.

Lemma 2.6. *There is a bijection between the cosets of E_{u_0} in S_f and the connected components of $\mathcal{D}_0$ given by*

$$g\mathrm{E}_{u_0} \mapsto g(\mathcal{D}_{u_0}).$$

Proof. It suffices to note that we have equality of cosets $g_1\mathrm{E}_{u_0} = g_2\mathrm{E}_{u_0}$ if and only if $g_1(u_0)$ and $g_2(u_0)$ are in the same connected component of $\mathcal{D}_0$. We have $g_1\mathrm{E}_{u_0} = g_2\mathrm{E}_{u_0}$ if and only if $g_1^{-1}g_2 \in \mathrm{E}_{u_0}$ or equivalently $g_1^{-1}g_2(\mathcal{D}_{u_0}) = \mathcal{D}_{u_0}$. Multiplying by g_1 gives the equivalent statement

$$g_1(\mathcal{D}_{u_0}) = g_1(g_1^{-1}g_2)(\mathcal{D}_{u_0}) = g_2(\mathcal{D}_{u_0}).$$

□

2.10. **Equalizers and fundamental groups.** We study the groups S_f and E_{u_0} using the language of equalizers. In general, given two maps

$$\xi_1, \xi_2 : (\mathcal{X}, x_0) \to (\mathcal{Y}, y_0)$$

between connected finite CW-complexes we define the *equalizer group* by

$$\mathrm{S}(\xi_1, \xi_2) := \{\gamma \in \pi_1(\mathcal{X}, x_0) \mid (\xi_1)_*(\gamma) = (\xi_2)_*(\gamma)\}.$$

Let $\iota : (\mathcal{E}\mathrm{q}(\xi_1, \xi_2), x_0) \hookrightarrow (\mathcal{X}, x_0)$ be the inclusion, and define

$$\mathrm{E}(\xi_1, \xi_2) := \iota_*(\pi_1(\mathcal{E}\mathrm{q}(\xi_1, \xi_2), x_0)).$$

We have $\mathrm{E}(\xi_1, \xi_2) \subseteq \mathrm{S}(\xi_1, \xi_2)$.

To study E_{u_0} and S_f we recall the natural maps $\rho_1, \rho_2 : (\mathcal{W}_f, v_0) \to (\mathcal{M}_A, m_0)$, and consider the groups $\mathrm{S}(\rho_1, \rho_2)$ and $\mathrm{E}(\rho_1, \rho_2)$. These are related to S_f and E_{u_0} via the defining map

$$\mathfrak{h} : \pi_1(\mathcal{W}_f, v_0) \to \mathrm{L}_f$$

for the covering $\omega : (\mathcal{U}_f, u_0) \to (\mathcal{W}_f, v_0)$. Recall that $\mathfrak{h}$ is defined by $\mathfrak{h}(\gamma) := [h_\gamma]$, where $[h_\gamma]$ is the unique covering automorphism taking u_0 to the endpoint of the lift of γ to $\mathcal{U}_f$ based at u_0.

Let $\mathfrak{h}_{\mathrm{S}}$ and $\mathfrak{h}_{\mathrm{E}}$ be the restrictions of $\mathfrak{h}$ to the subgroups $\mathrm{S}(\rho_1, \rho_2)$ and $\mathrm{E}(\rho_1, \rho_2)$ in $\pi_1(\mathcal{W}_f, v_0)$. Then we have the following.

Proposition 2.7. *The maps $\mathfrak{h}_{\mathrm{E}}$ and $\mathfrak{h}_{\mathrm{S}}$ define isomorphisms*

$$\mathfrak{h}_{\mathrm{E}} : \mathrm{E}(\rho_1, \rho_2) \to \mathrm{E}_{u_0},$$

and

$$\mathfrak{h}_{\mathrm{S}} : \mathrm{S}(\rho_1, \rho_2) \to \mathrm{S}_f.$$

Lemma 2.6 gives the following result.

Corollary 2.8. *There is a bijection between the connected components of $\mathcal{D}_0$ and the cosets* $\mathrm{S}(\rho_1, \rho_2)/\mathrm{E}(\rho_1, \rho_2)$.

Remark 2.9. Since the spaces $\mathcal{U}_f$ and $\mathcal{W}_f$ are connected, $\mathrm{S}(\rho_1, \rho_2)$ only depends on f and not on the choice of basepoint u_0. It is currently unknown, at least to the authors, whether $\mathcal{V}_f$ is always connected, or whether $\mathrm{E}(\rho_1, \rho_2)$ is independent of the choice of basepoint.

2.11. **Example.** We apply the previous definitions to a specific example that will be useful in the next section.

Let

$$\mathcal{X} := \{(x, y) \in \mathbb{C}^2 \mid x \neq 0, x \neq 1, y \neq 0, y \neq 1 \text{ and } x + y \neq 1\},$$

and let ξ_1, ξ_2 be the projections onto the x and y coordinates. Let $\delta : \mathcal{X} \to \mathcal{X}$ be given by $\delta(x, y) := (y, x)$. Then $\xi_1 = \xi_2 \circ \delta$ and $\xi_2 = \xi_1 \circ \delta$.

Define a quotient space of $\mathcal{X}$ as follows. Let

$$\mathcal{Q} := \{(z, w) \in \mathbb{C}^2 \mid z \neq 1,\ w \neq 0 \text{ and } w - z \neq -1\}.$$

Let a satisfy $1/2 < a < 1$, let $x_0 := (a, a) \in \mathcal{E} := \mathcal{E}\mathrm{q}(\xi_1, \xi_2)$, and let $q_0 := (2a, a^2)$. Let $\mathfrak{s}$ be the quotient map

$$\begin{array}{rcl} \mathfrak{s} : (\mathcal{X}, x_0) & \to & (\mathcal{Q}, q_0) \\ (x, y) & \mapsto & (x + y, xy). \end{array}$$

We now describe the fundamental group of $\mathcal{Q}$.

Let $\mathcal{K}_1 \subseteq \mathcal{X}$ be

$$\mathcal{K}_1 := \xi_1^{-1}(a) = \{(a, y) \in \mathcal{X} \mid y \notin \{0, 1 - a, 1\}\},$$

and let $\mathcal{K}_2 := \delta(\mathcal{K}_1)$. Since $1/2 < a < 1$, the inclusions of $\mathcal{K}_1$ and $\mathcal{K}_2$ into $\mathcal{X}$ are general fibers of the singular fibrations ξ_1 and ξ_2. Thus, the inclusions induce epimorphisms

$$\pi_1(\mathcal{K}_1, x_0) \to \ker((\xi_1)_*) \quad \text{and} \quad \pi_1(\mathcal{K}_2, x_0) \to \ker((\xi_2)_*).$$

Let

$$\mathcal{L} := \{(z, w) \in \mathcal{Q} \mid w = a(z - a)\},$$

which is the image of $\mathcal{K}_1$ and $\mathcal{K}_2$ inside $\mathcal{Q}$, and let

$$\overline{\mathcal{E}} := \{(z, w) \in \mathcal{Q} \mid 4w = z^2\},$$

which is the image of $\mathcal{E}$ inside $\mathcal{Q}$. Then $q_0 \in \mathcal{L} \cap \overline{\mathcal{E}}$ and the line $\mathcal{L}$ is tangent to the conic $\overline{\mathcal{E}}$. Let $\iota_{\mathcal{L}} : \mathcal{L} \to \mathcal{Q}$ be the inclusion map. Since $\mathcal{L}$ is isomorphic to a complex line with three points removed, its fundamental group $\pi_1(\mathcal{L}, q_0)$ is the free group on three generators r', s' and c', where

(a) r', c' and s' are freely homotopic to small positively-oriented loops around the three punctures of $\mathcal{L}$ corresponding to $\{w = 0\}$, $\{z = 1\}$ and $\{w = z - 1\}$, and
(b) $r'c's'$, $c's'r'$ and $s'r'c'$, are each homotopic rel q_0 to a large simple closed loop encircling the three punctures.

By the Zariski-van Kampen theorem, the map

$$(\iota_{\mathcal{L}})_* : \pi_1(\mathcal{L}, q_0) \to \pi_1(\mathcal{Q}, q_0)$$

is surjective. Thus, $r := (\iota_{\mathcal{L}})_*(r')$, $s := (\iota_{\mathcal{L}})_*(s')$ and $c := (\iota_{\mathcal{L}})_*(c')$ generate $\pi_1(\mathcal{Q}, q_0)$.

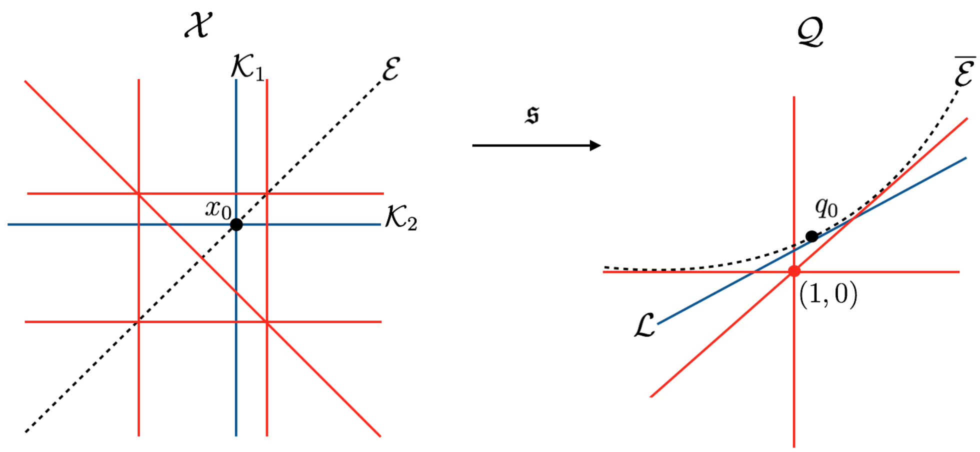

Figure 1. The space $\mathcal{X}$ (on the left) is the complement of the vertical, horizontal and anti-diagonal lines. The space $\mathcal{Q}$ (on the right) it is the complement in $\mathbb{C}^2$ of the three lines intersecting at $(1, 0)$.

Since the lines $\{w = 0\}$, $\{w = z - 1\}$ and $\{z = 1\}$ all meet at the point $(1, 0)$, the pencil of lines on $\mathcal{Q}$ through $(1, 0)$ defines a fiber bundle

$$\begin{aligned} \beta : (\mathcal{Q}, q_0) &\to (\mathbb{P}^1 - \mathcal{R}, r_0) \\ (z, w) &\mapsto [z - 1 : w], \end{aligned}$$

where $\mathcal{R} := \{[0 : 1], [1 : 0], [1 : 1]\} \subseteq \mathbb{P}^1$, and $r_0 := [2a - 1 : a^2]$. The general fiber $\mathcal{F}$ is isomorphic to the punctured plane $\mathbb{C} - \{0\}$, and the monodromy of the fibration preserves the complex structure of $\mathbb{C} - \{0\}$, and hence acts trivially on the fundamental group of the fiber $\mathcal{F}_0$ through q_0. Thus, $\pi_1(\mathcal{Q}, q_0)$ is isomorphic to $\mathbb{Z} \times \mathbb{F}_2$.

Let d' be the positively-oriented generator of $\pi_1(\mathcal{F}_0, q_0)$ and d its image in $\pi_1(\mathcal{Q}, q_0)$. Let $\overline{r} := \beta_*(r)$, $\overline{s} := \beta_*(s)$ and $\overline{c} := \beta_*(c)$ Then $\overline{rcs}$, $\overline{csr}$ and $\overline{src}$ are represented by loops that encircle $\mathcal{R}$ in $\mathbb{P}^1$ and hence are trivial in $\pi_1(\mathbb{P}^1 - \mathcal{R}, r_0)$. It follows that the exact sequence of the fiber bundle gives the commutative diagram:

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(\mathcal{F}_0, q_0) \arrow[r] \arrow[d] & \pi_1(\mathcal{Q}, q_0) \arrow[r, "\beta_*"] \arrow[d] & \pi_1(\mathbb{P}^1 - \mathcal{R}, r_0) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \langle d' \rangle \arrow[r] & \langle r, c, s, d \mid d = rcs = csr = src \rangle \arrow[r] & \langle \overline{r}, \overline{c}, \overline{s} \mid \overline{rcs} = 1 \rangle \arrow[r] & 1
\end{tikzcd}$$

where the vertical arrows are isomorphisms, and the horizontal arrows are exact. We have shown the following.

Lemma 2.10. *The fundamental group of $\mathcal{Q}$ is given by*

$$\pi_1(\mathcal{Q}, q_0) = \langle r, c, s, d \mid d = rcs = csr = src \rangle \simeq \mathbb{Z} \times \mathbb{F}_2,$$

and, in particular, the element $d \in \pi_1(\mathcal{Q}, q_0)$ generates the central $\mathbb{Z}$ factor.

The next proposition will be a key step in our proof of Theorem 1.3.

Proposition 2.11. *There is a $\gamma \in \mathrm{S}(\xi_1, \xi_2)$ so that $\gamma^n \notin \mathrm{E}(\xi_1, \xi_2)$ for all $n \neq 0$, and hence*

$$[\mathrm{S}(\xi_1, \xi_2) : \mathrm{E}(\xi_1, \xi_2)] = \infty.$$

Proof. We prove the proposition in two steps.

Step 1. We show that the image of $\mathrm{E}(\xi_1, \xi_2)$ under $\mathfrak{s}_*$ intersects $\ker(\beta_*)$ in the trivial element.

The map $\mathfrak{s}$ sends $\mathcal{E}$ isomorphically onto $\overline{\mathcal{E}}$ which is tangent to the lines $\{w = 0\}$ and $\{w = z - 1\}$ (see Figure 1).

Let $j : (\overline{\mathcal{E}}, q_0) \to (\mathcal{Q}, q_0)$, be the inclusion map. On $\overline{\mathcal{E}}$, the pencil β defines a covering

$$\beta \circ j : (\overline{\mathcal{E}}, q_0) \to (\mathbb{P}^1 - \mathcal{R}, r_0),$$

since all lines in $\mathcal{Q}$ through $(1, 0)$ intersect $\overline{\mathcal{E}}$ transversally in 2 points, and we have an endomorphism

$$(\beta \circ j)_* : \pi_1(\overline{\mathcal{E}}, q_0) \to \pi_1(\mathbb{P}^1 - \mathcal{R}, r_0).$$

Thus, $j_*(\pi_1(\overline{\mathcal{E}}, q_0))$ intersects the kernel of β_* in the identity element.

Step 2. We find an element $\gamma \in \mathrm{S}(\xi_1, \xi_2)$ of infinite order, such that $\mathfrak{s}_*(\gamma^n)$ lies in the kernel of β_* for all $n \neq 0$.

Let $\gamma_1 \in \pi_1(\mathcal{X}, x_0)$ be freely homotopic to a simple closed curve ℓ on $\mathcal{K}_1$ that goes around the image of the path

$$\begin{aligned} \tau : [0,1] &\to \mathbb{C}^2 \\ t &\mapsto (a, t) \end{aligned}$$

with positive orientation relative to the complex structure. The image of τ is the straight line from $(a, 0)$ to $(a, 1)$. Thus, γ_1 may also be thought of as freely homotopic in $\mathcal{K}_1$ to a loop around the point at infinity on $\mathcal{K}_1$. Let $\overline{\tau}$ be the path given by

$$\begin{aligned} \overline{\tau} : [0,1] &\to \mathbb{P}^1 \\ t &\mapsto (\beta \circ \mathfrak{s} \circ \tau)(t) = [a + t - 1 : at]. \end{aligned}$$

This is a path from $[1 : 0]$ to $[1 : 1]$ that passes through $[0 : 1]$ (when $t = 1 - a$). It follows that $(\mathbb{P}^1 - \mathcal{R}) - \overline{\tau}([0, 1])$ is simply connected. Since $\beta_*(\mathfrak{s}_*(\gamma_1))$ lies in $(\mathbb{P}^1 - \mathcal{R}) - \overline{\tau}([0, 1])$, $\mathfrak{s}_*(\gamma_1)$ lies in the kernel of β_*.

Let $\gamma_2 := \delta_*(\gamma_1) \in \pi_1(\mathcal{X}, x_0)$, and let $\gamma := \gamma_1\gamma_2$. Since $(\xi_1)_*(\gamma) = (\xi_2)_*(\gamma)$, we have $\gamma \in \mathrm{S}(\xi_1, \xi_2)$. Furthermore, $\mathfrak{s}_*(\gamma^n) = \mathfrak{s}_*(\gamma_1)^{2n} = \mathfrak{s}_*(\gamma_2)^{2n}$ and lies in the kernel of β_* as desired.

To see that $\mathfrak{s}_*(\gamma)$ has infinite order in $\pi_1(\mathcal{Q}, q_0)$, note that $\mathcal{L} = \mathfrak{s}(\mathcal{K}_1)$ is a generic line in $\mathcal{Q}$. Continuously moving $\mathcal{L}$ along lines of the same slope to a generic line through $(1, 0)$ in $\mathcal{Q}$ defines a free homotopy equivalence of $\mathfrak{s}_*(\gamma)$ to a loop in a fiber of β. Since $\mathfrak{s}_*(\gamma_1)$ generates $\pi_1(\mathcal{L} - (\mathfrak{s} \circ \tau)([0, 1]), q_0)$, we have $\mathfrak{s}_*(\gamma_1) = d$. Thus $\mathfrak{s}_*(\gamma) = d^2$, which has infinite order in $\pi_1(\mathcal{Q}, q_0)$. $\square$

Remark 2.12. Since γ maps to the center of $\pi_1(\mathcal{Q}, q_0)$ under the map $\mathfrak{s}_*$, we could have replaced γ by any conjugate of γ, and the statement of Proposition 2.11 would still hold.

3. Application to quadratic rational maps

Let f represent an element of $\mathrm{Per}_4(0)^*$. By conjugating with a Möbius transformation, we may suppose that f has a superattracting cycle of the form

$$\begin{tikzcd} 0 \arrow[r, "2"] & \infty \arrow[r] & 1 \arrow[r] & a \arrow[lll, bend left] \end{tikzcd} \tag{1}$$

where 0 is the periodic critical point.

Let $A := \{0, 1, \infty, a\}$, let $B := A \cup \{b\}$ where $b \notin A$ is the other critical value of f, and let $u_0 \in \mathcal{D}_f$ be the basepoint associated to the rational map f.

Let (α, β, F) represent a point in $\mathcal{W}_f$. By post-composing with Möbius transformations, we may suppose that

$$\alpha|_{\{0,1,\infty\}} = \mathrm{id}|_{\{0,1,\infty\}} \quad \text{and} \quad \beta|_{\{0,1,\infty\}} = \mathrm{id}|_{\{0,1,\infty\}}.$$

Then the point $[\alpha, \beta, F] \in \mathcal{W}_f$ is determined by the complex numbers $x := \alpha(a)$, $y := \beta(a)$, $z := \beta(b)$, and a quadratic rational map

$$F : (\mathbb{P}^1, \{0, 1, \infty, x\}) \to (\mathbb{P}^1, \{0, 1, \infty, y, z\})$$

satisfying

$$\begin{tikzcd} 0 \arrow[d, "2"] & \infty \arrow[d] & 1 \arrow[d] & x \arrow[d] & c \arrow[d, "2"] \\ \infty & 1 & y & 0 & z \end{tikzcd}$$

where 0 and c are the two critical points of F. The map F must be of the form

$$F(t) = \frac{(t - x)(t - r)}{t^2}$$

where $r = \frac{y}{x-1} + 1$. Thus, F and the critical value z are determined by x and y, as long as x, y satisfy certain algebraic conditions. In these coordinates, the map $\rho : \mathcal{W}_f \to \mathcal{M}_A \times \mathcal{M}_A$ is given by

$$\rho = (\rho_1, \rho_2) : [x, (y, z), F] \mapsto (x, y),$$

and the image of ρ is equal to $\mathcal{M}_A \times \mathcal{M}_A - \mathcal{C}$ for an algebraic set $\mathcal{C}$. A computation shows that $\mathcal{C}$ is given explicitly by

$$\mathcal{C} = \{x + y = 1\} \cup \{x^2 - y - 2x + 1 = 0\} \cup \{x^2 + y = 1\} \cup \{2xy + x^2 - y - 2x + 1 = 0\}.$$

We have shown the following.

Proposition 3.1. *The map $\rho : \mathcal{W}_f \to \mathcal{M}_A \times \mathcal{M}_A$ is an isomorphism onto a Zariski dense subset, and $\mathcal{V}_f$ maps isomorphically to $\mathcal{W}_f \cap \mathcal{E}\mathrm{q}(\rho_1, \rho_2)$; in particular, $\mathcal{V}_f$ is connected.*

We are now ready to complete the proof of Theorem 1.3.

Proof of Theorem 1.3. By Corollary 2.8, to prove Theorem 1.3 it suffices to show that there is an element $\gamma \in \mathrm{S}(\rho_1, \rho_2)$ such that no nonzero power of γ lies in $\mathrm{E}(\rho_1, \rho_2)$. We do this by reducing to the setting of the example in Section 2.11.

The space $\mathcal{W}_f$ embeds onto a Zariski dense subset in $\mathcal{X} := \mathcal{M}_A \times \mathcal{M}_A - \{x + y = 1\}$. Let $\iota : \mathcal{W}_f \to \mathcal{X}$ be the embedding, and let $x_0 := \iota(v_0)$. Then the induced map

$$\iota_* : \pi_1(\mathcal{W}_f, v_0) \to \pi_1(\mathcal{X}, x_0)$$

is an epimorphism. Furthermore, the maps $\rho_1, \rho_2 : \mathcal{W}_f \to \mathcal{M}_A$ factor as

$$\rho_1 = \xi_1 \circ \iota \quad \text{and} \quad \rho_2 = \xi_2 \circ \iota$$

where $\xi_1, \xi_2 : \mathcal{M}_A \times \mathcal{M}_A \to \mathcal{M}_A$ are projections onto the x and y coordinates. It follows that $\iota_*(\mathrm{S}(\rho_1, \rho_2)) = \mathrm{S}(\xi_1, \xi_2)$ and $\iota_*(\mathrm{E}(\rho_1, \rho_2)) = \mathrm{E}(\xi_1, \xi_2)$.

By Proposition 2.11 the index $[\mathrm{S}(\xi_1, \xi_2) : \mathrm{E}(\xi_1, \xi_2)]$ is infinite. It follows that the index $[\mathrm{S}(\rho_1, \rho_2) : \mathrm{E}(\rho_1, \rho_2)]$ is also infinite. By Lemma 2.6 this implies that $\mathcal{D}_f$ has infinitely many components. $\square$

We finish with a constructive description of an element in S_f whose action on connected components of $\mathcal{D}_f$ has an infinite orbit. The Birman exact sequence [B] for $A = \{0, 1, \infty, a\}$ is

$$1 \to \pi_1(\mathbb{P}^1 - \{0, 1, \infty\}, a) \xrightarrow{\eta} \mathrm{Mod}_A \longrightarrow \mathrm{Mod}_{\{0,1,\infty\}} \to 1,$$

where η takes a loop ℓ based at a to the point-push map associated to ℓ. Since $\mathrm{Mod}_{\{0,1,\infty\}}$ is trivial, η is an isomorphism. Our choice of basepoint $a = m_0$ identifies $\pi_1(\mathbb{P}^1 - \{0, 1, \infty\}, a)$ with $\pi_1(\mathcal{M}_A, m_0)$ and η becomes the defining map from $\pi_1(\mathcal{M}_A, m_0)$ to Mod_A that determines the regular covering $\mathcal{T}_A \to \mathcal{M}_A$.

Proposition 3.2. *Let $\mathcal{D}_{u_0}$ be the connected component of $\mathcal{D}_f$ containing u_0. Let $\kappa \in \pi_1(\mathbb{P}^1 - \{0, 1, \infty\}, a)$ be represented by a simple closed path based at a separating $\{0, 1\}$ from $\{\infty\}$. Then there is an element $s \in \mathrm{S}_f$ such that*

$$\Phi_f(s) = s_A = \eta(\kappa),$$

and the map

$$n \mapsto s^n(\mathcal{D}_{u_0}),$$

defines a bijection from $\mathbb{Z}$ to a subset of the connected components of $\mathcal{D}_f$.

Proof. Let $\gamma_1' \in \pi_1(\mathcal{X}, x_0)$ be such that $(\xi_1)_*(\gamma_1') = \kappa$. As in our proof of Proposition 2.11, we can assume that γ_1' has a representative ℓ contained in the generic fiber $\mathcal{K}_1 := \xi_1^{-1}(a)$, such that ℓ avoids the set

$$\tau := \{(a, y) \in \mathbb{C}^2 \mid 0 \le y \le 1\}$$

in $\mathcal{K}_1$. Let $\gamma_1 \in \pi_1(\mathcal{W}_f, v_0)$ be such that the map induced by inclusion $\iota : \mathcal{W}_f \to \mathcal{X}$ gives

$$\iota_*(\gamma_1) = \gamma_1',$$

and let $\gamma_2 \in \pi_1(\mathcal{W}_f, v_0)$ be such that

$$\iota_*(\gamma_2) = \delta_*(\gamma_1').$$

The elements γ_1 and γ_2 exist because ι_* is surjective. Then $\gamma = \gamma_1\gamma_2$ defines an element of $\mathrm{S}(\rho_1, \rho_2)$, and $s := \mathfrak{h}(\gamma) \in \mathrm{S}_f$ satisfies $\Phi_f(s) = s_A = \eta(\kappa)$. $\square$

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