

Small dilatation pseudo-Anosov mapping classes coming from the simplest hyperbolic braid.

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Abstract

In this paper we study the minimum dilatation pseudo-Anosov mapping classes coming from fibrations over the circle of a single 3-manifold, namely the mapping torus for the "simplest pseudo-Anosov braid". The dilatations that arise include the minimum dilatations for orientable mapping classes for genus $g = 2, 3, 4, 5, 8$ as well as Lanneau and Thiffeault's conjectural minima for orientable mapping classes, when $g = 2, 4 \pmod{6}$. Our examples also show that the minimum dilatation for orientable mapping classes is strictly greater than the minimum dilatation for non-orientable ones when $g = 4, 6, 8$.

1 Introduction

Let S_g be a closed oriented surface of genus $g \geq 1$, and let Mod_g be the *mapping class group*, that is, the group of orientation preserving automorphisms of S_g up to isotopy. A mapping class $\phi \in \text{Mod}_g$ is called *pseudo-Anosov* if S_g has a pair of ϕ -invariant, transversally measured, singular foliations on which ϕ acts by stretching along one and contracting along the other by a constant $\delta(\phi) > 1$. The constant $\delta(\phi)$ is called the (*geometric*) *dilatation* of ϕ . A mapping class is pseudo-Anosov if it is neither periodic nor reducible [Thur88] [FLP79] [CB88].

A pseudo-Anosov mapping class ϕ is defined to be *orientable* if its invariant foliations are orientable. Let $\delta_{\text{hom}}(\phi)$ be the spectral radius of the action of ϕ on the first homology of S . Then

$$\delta_{\text{hom}}(\phi) \leq \delta(\phi),$$

with equality if and only if ϕ is orientable (see, for example, [LT09] p. 5).

The dilatations $\delta(\phi)$ satisfy reciprocal monic integer polynomials of degree bounded from above by $6g - 6$ [Thur88]. If ϕ is orientable the degree is bounded by $2g$. For fixed g , it follows that $\delta(\phi)$ achieves a minimum $\delta_g > 1$ in Mod_g (cf. [AY80] [Iva90]). Let δ_g^+ be the minimum dilatation for orientable pseudo-Anosov elements of Mod_g .

In this paper, we address the question:

Question 1.1 *What is the behavior of δ_g and δ_g^+ as functions of g ?*

So far, exact values of δ_g have only been found for $g \leq 2$. For $g = 1$, $\text{Mod}_1 = \text{SL}(2; \mathbb{Z})$, and we have

$$\delta_1 = \delta_1^+ = \frac{3 + \sqrt{5}}{2}.$$

For $g = 2$, Cho and Ham [CH08] show that δ_2 is the largest real root of

$$x^4 - x^3 - x^2 - x + 1 = 0$$

or approximately 1.72208.

In the orientable case more is known due to recent results of Lanneau and Thiffeault [LT09]. Given $(a, b) \in \mathbb{Z} \oplus \mathbb{Z}$ with $0 < a < b$, let

$$LT_{(a,b)}(t) = t^{2b} - t^b(1 + x^a + x^{-a}) + 1,$$

and let $\lambda_{(a,b)}$ be the largest real root of $LT_{(a,b)}(t)$.

Theorem 1.2 (Lanneau-Thiffeault [LT09] Thm. 1.2, Thm. 1.3) For $g = 2, 3, 4, 6, 8$,

$$\lambda_{(1,g)} \leq \delta_g^+$$

with equality when $g = 2, 3, 4$.

For $g = 2$, the value of δ_2^+ was first determined by Zhironov [Zhi95]. For $g = 5$, Lanneau and Thiffeault show that δ_5^+ equals Lehmer's number, which is realized by an example of Leininger [Lein04]. Lanneau and Thiffeault also find a lower bound for δ_7^+ , but so far no one has found an example with that dilatation.

Based on their calculations, Lanneau and Thiffeault ask: *is $\delta_g^+ = \lambda_g$ for even g ?* We call the affirmative answer to their question the LT-conjecture.

In our first result, we improve on the following previously known best bounds for minimum dilatation of infinite families

$$(\delta_g)^g \leq (\delta_g^+)^g \leq 2 + \sqrt{3}$$

(see [Min06] [HK06]).

Theorem 1.3 If $g = 0, 1, 3, 4 \pmod{6}$, $g \geq 3$, then

$$\delta_g \leq \lambda_{(3,g+1)},$$

and if $g = 2, 5 \pmod{6}$ and $g \geq 5$, then

$$\delta_g \leq \lambda_{(1,g+1)}.$$

For the orientable case, our results complement those of Lanneau and Thiffeault for $g = 2, 4 \pmod{6}$.

Theorem 1.4 If $g = 1, 3 \pmod{6}$,

$$\delta_g^+ \leq \lambda_{(3,g+1)};$$

if $g = 2, 4 \pmod{6}$,

$$\delta_g^+ \leq \lambda_{(1,g)};$$

and if $g = 5 \pmod{6}$,

$$\delta_g^+ \leq \lambda_{(1,g+1)}.$$

Putting Theorem 1.4 together with Lanneau and Thiffeault's lower bound for $g = 8$, we prove the LT-conjecture for $g = 8$.

Corollary 1.5 For $g = 8$, we have

$$\delta_8^+ = \lambda_{(1,8)}.$$

For large g , it is known that δ_g and δ_g^+ converges to 1. Furthermore, we have

$$\log(\delta_g) \asymp \frac{1}{g} \quad \text{and} \quad \log(\delta_g^+) \asymp \frac{1}{g} \quad (1)$$

(see [Pen91] [McM1] [Min06] [HK06]). The LT-conjecture together with (1) leads to the natural question:

Question 1.6 (e.g., [McM1], p.551, [Farb06], Problem 7.1) *Do the sequences*

$$(\delta_g)^g \quad \text{and} \quad (\delta_g^+)^g$$

converge as g grows? What is the limit?

The examples in this paper show the following.

Proposition 1.7 *If the limit exists, then*

$$\limsup_{g \rightarrow \infty} (\delta_g)^g \leq \frac{3 + \sqrt{5}}{2}.$$

If the LT-conjecture is true, then δ_{2m}^+ is a monotone strictly decreasing sequence (see Proposition 4.6) that converges to $\frac{3+\sqrt{5}}{2}$. Thus, the LT-conjecture implies equality in Proposition 1.7.

Lanneau and Thiffeault show that $\delta_5^+ \leq \delta_6^+$, and hence δ_g^+ is not strictly monotone decreasing (cf. [Farb06] Question 7.2). Theorem 1.4 shows the stronger statement.

Proposition 1.8 *If the LT-conjecture is true, then $\delta_g^+ \leq \delta_{g+1}^+$, whenever $g = 5 \pmod{6}$.*

Another example concerns the question of whether the inequality $\delta_g \leq \delta_g^+$ is strict for any or all g . Table 1 shows the following.

Proposition 1.9 *For $g = 4, 6, 8$ we have*

$$\delta_g < \delta_g^+.$$

If the LT conjecture is true, then Theorem 1.3 and Proposition 4.6 imply that the phenomena revealed in Proposition 1.9 repeats itself periodically.

Proposition 1.10 *If the LT-conjecture is true, then for all even $g \geq 4$ we have*

$$\delta_g < \delta_g^+.$$

We prove Theorem 1.3 and Theorem 1.4 using a family of mapping classes $\phi_{(a,b)}$ that come from a fibered face of a single 3-manifold M . This is interesting in light of the *Universal Finiteness Theorem* due to Farb, Leininger and Margalit [FLM09]. For any pseudo-Anosov mapping class $\phi \in \text{Mod}_g$, let $M(\phi)$ be the mapping torus of ϕ after removing tubular neighborhoods of suspensions of the singularities. Let

$$\mathcal{T}_P = \{M(\phi) : \lambda(\phi) \leq P^g\}.$$

Then \mathcal{T}_P is a finite set for all P ([FLM09] Thm. 1.1). The asymptotic equations (1) imply that

$$\mathcal{T} = \{M(\phi) : \phi \in \text{Mod}_g, \phi \text{ pseudo-Anosov}, \lambda(\phi) = \delta_g\}$$

and

$$\mathcal{T}^+ = \{M(\phi) : \phi \in \text{Mod}_g, \phi \text{ pseudo-Anosov}, \lambda(\phi) = \delta_g^+\}$$

are finite. For our examples, M is the complement of a two component link L , known as 6_2^2 in Rolfsen's table [Rol76]. Dilatations of mapping classes coming from another 3-manifold can be found in [KT08].

The following is a table of the minimal dilatations that arise in this paper's examples for genus 1 through 12. All numbers in the table are truncated to 5 decimals. An asterisk * marks the numbers that have been verified to equal δ_g^+ (resp. δ_g). For singularity-type, we use the convention that (a_1, \dots, a_k) means that the singularities of the invariant foliations have degrees a_1, \dots, a_k (see Lanneau and Thiffeault's notation [LT09], p.3). The singularity-types for our examples are derived from the formula given in Proposition 3.5.

g	orientable	degrees of singularities	unconstrained	degrees of singularities
1	2.61803*	no sing.	2.61803*	no sing.
2	1.72208*	(4)	1.72208*	(4)
3	1.40127*	(2, 2, 2, 2)	1.40127	(2,2,2,2)
4	1.28064*	(10,2)	1.26123	(3,3,3,3)
5	1.17628*	(16)	1.17628	(16)
6	-	-	1.1617	(5,5,5,5)
7	1.13694	(6,6,6,6)	1.13694	(6,6,6,6)
8	1.12876*	(22,6)	1.1135	(25,1,1,1)
9	1.1054	(8,8,8,8)	1.1054	(8,8,8,8)
10	1.10149	(28,8)	1.09466	(9,9,9,9)
11	1.08377	(34,2,2,2)	1.08377	(34,2,2,2)
12	-	-	1.07874	(11,11,11,11)

Table 1: Minimal orientable and unconstrained dilatations coming from M

For $g = 1, 2, 3, 4, 5$, our orientable examples agree both in dilatation and in singularity-type with previously found minimizing examples. Thus, for example, we have shown that

$$M \in \mathcal{T} \cap \mathcal{T}^+.$$

For $g = 8$, it agrees with the singularity-type anticipated by Lanneau and Thiffeault (see [LT09]). For $g = 6k$, we do not get any orientable examples out of fibrations of M , and for $g = 7$, our minimal example gives a strictly larger dilatation than Lanneau and Thiffeault's lower bound.

Section 2 contains a brief review of Thurston norms, Alexander norms, and the Teichmüller polynomial. These are the basic tools used in this paper. In Section 3 we describe our family of examples, and in Section 4 we prove Theorem 1.3 and Theorem 1.4.

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2 Background and tools

We give a brief review of fibrations of a hyperbolic 3-manifold M and their invariants, emphasizing tools that we will use in the rest of the paper. For more details see, for example, [Thur86] [FLP79] [McM1] [McM02].

The theory of fibered faces of the Thurston norm ball and Teichmüller polynomials provides a way to map out the possible fibrations of a given hyperbolic manifold and the dilatations of their monodromies in a single picture. Assume M is a compact hyperbolic 3-manifold with boundary. Given an embedded surface S on M , let $\chi_-(S)$ be the sum of $|\chi(S_i)|$, where S_i are the irreducible components of S with negative Euler characteristic. The Thurston norm of ψ is defined to be

$$\|\psi\|_T = \min \chi_-(S)$$

where the minimum is taken over oriented embedded surfaces $(S, \partial S) \subset (M, \partial M)$ such that the class of S in $H_2(M, \partial M; \mathbb{Z})$ is dual to ψ

Elements of $H^1(M; \mathbb{Z})$ are canonically associated with epimorphisms

$$\pi_1(M; \mathbb{Z}) \rightarrow \mathbb{Z}$$

which factor through epimorphisms

$$H_1(M; \mathbb{Z}) \rightarrow \mathbb{Z}.$$

Thus, we have a lattice $\Lambda_M \subset \mathbb{R}^{b_1(M)}$ equal to any of the following naturally identified objects.

$$H^1(M; \mathbb{Z}) = \text{Hom}(\pi_1(M) \rightarrow \mathbb{Z}) = \text{Hom}(H_1(M; \mathbb{Z}), \mathbb{Z}).$$

If $\psi \in \Lambda_M$ is induced by a fibration

$$M \rightarrow S^1$$

we say that ψ is *fibered*. In this case,

$$\|\psi\|_T = \chi_-(S),$$

where S is the fiber of ψ . The *monodromy* of ψ is the mapping class $\phi : S \rightarrow S$, such that M is the mapping torus of ϕ , and ψ is the natural projection to S^1 . Since M is hyperbolic, ϕ is automatically pseudo-Anosov.

Let Σ be the unit sphere in $\mathbb{R}^{b_1(M)}$ with respect to the extended Thurston norm. Then Σ is a polyhedron and Λ_M projects to a dense subset of Σ , called the *rational points* of Σ . The fibered elements of Λ_M project to the (open) faces of Σ . If a face of Σ contains the projection of a fibered element, then all elements that project to the same face must be fibered. A face of Σ containing the projection of a fibered element is thus called a *fibered face* of Σ .

Let $\psi \in \Lambda_M$ be a fibered element, and let ψ_0 be the element of Λ_M that lies closest to the origin along the ray containing ψ . Then

$$\psi = r(\psi_0),$$

for some positive integer r , and the fibration associated to ψ is obtained by taking the fibration associated to ψ_0 and composing with the r -cyclic covering of S^1 . It follows that ψ_0 has connected fibers, while the fibers of ψ have r -connected components. The dilatation of ϕ is given by

$$\delta(\phi) = \delta(\phi_0)^{1/r},$$

while $\|\psi\|_T = r\|\psi_0\|_T$.

Theorem 2.1 ([Fri82], Theorem E) *There is a continuous function $\mathcal{Y}(\psi)$ defined on the entire fibered cone in $\mathbb{R}^{b_1(M)}$, so that if ψ is fibered with monodromy ϕ , then*

$$\mathcal{Y}(\psi) = \frac{1}{\log(\delta(\phi))}.$$

The function \mathcal{Y} is homogeneous of degree one, and is a concave function tending to zero along the boundary of the cone.

The Alexander polynomial Δ_M of M is a polynomial in $\mathbb{Z}[G]$, where $G = H_1(M; \mathbb{Z})$. Each element $\psi \in \Lambda_M$ determines an epimorphism of \mathbb{Z} -modules:

$$\rho : \mathbb{Z}[G] \rightarrow \mathbb{Z}[t, t^{-1}],$$

where we identify $\mathbb{Z}[t]$ with the group ring over $\mathbb{Z} = H_1(S^1; \mathbb{Z})$. This defines a specialization

$$\Delta_{(M, \psi)} = \rho(\Delta_M) \in \mathbb{Z}[t].$$

The polynomial $\Delta_{(M, \psi)}$ is the characteristic polynomial for the monodromy ϕ of ψ acting on $H_1(S; \mathbb{Z})$, where S is the fiber of ψ . Thus, the degree of the Alexander polynomial specialized to a particular ψ is the rank of $H_1(S; \mathbb{Z})$. This is called the Alexander norm of ψ . The *homological dilatation* of ϕ , that is, the spectral radius of the action of ϕ on $H_1(S; \mathbb{Z})$, is the maximum among norms of roots of $\Delta_{(M, \psi)}$. We denote the homological dilatation by $\delta_{\text{hom}}(\phi)$.

The *Teichmüller polynomial* Θ associated to a fibered face of Σ_M is analogous to the Alexander polynomial. It is a polynomial defined on λ_M such that for each ψ in the cone over the fibered face, the geometric dilatation $\delta(\phi)$ of the monodromy is the largest real root of Θ specialized to ψ [McM1].

The following result will be used in Section 3.

Theorem 2.2 ([McM02] Thm. 7.1) *Let M be the complement in S^3 of a tubular neighborhood of a link L . If L has 8 or fewer crossings, then the Thurston and Alexander norms agree on $H^1(M; \mathbb{Z})$.*

3 The mapping torus for the simplest pseudo-Anosov braid

We now look at a particular 3-manifold, and study properties of its fibrations. Let $M = S^3 \setminus N(L)$, where L is the link drawn in two ways in Figure 1, and $N(L)$ is a tubular neighborhood. As seen

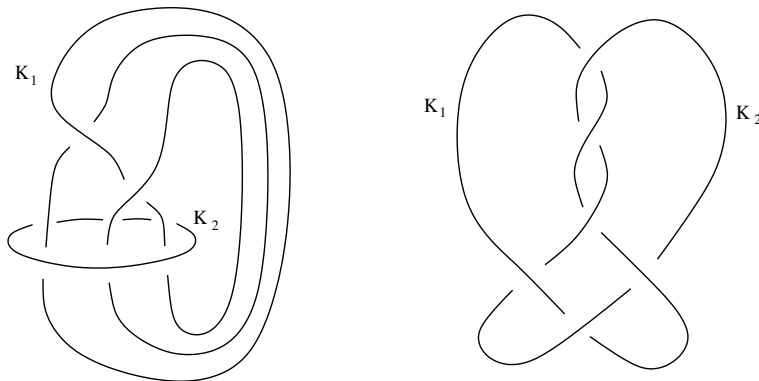


Figure 1: Two diagrams for the link 6_2^2 .

from the left diagram in Figure 1, M fibers over the circle with fiber a four times punctured sphere S . Let $\psi \in \Lambda_M$ be the induced element. Let K_1 be the component of L passing through S , and let K_2 be component of L that is one of the four boundary components of L .

The monodromy ϕ of ψ is the composition of two Dehn twists determined by 180 degree rotations as drawn in Figure 2, and has dilatation

$$\lambda(\phi) = \frac{3 + \sqrt{5}}{2}.$$

Its lift to a torus realizes δ_1 , and its dilatation is smallest possible for mapping classes defined on S making β the so-called “simplest pseudo-Anosov braid”.

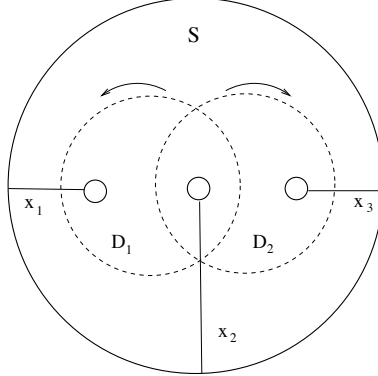


Figure 2: Braid monodromy associated to $\beta = \sigma_1 \sigma_2^{-1}$.

The Thurston norm and the Alexander norm both are given by

$$\|(a, b)\| = \max\{2|a|, 2|b|\}, \quad (2)$$

where $(a, b) \in H^1(M; \mathbb{Z})$ denotes the class that evaluates to a on the meridian μ_1 of K_1 and b on the meridian μ_2 of K_2 (see also [McM1] §11).

The lattice points Λ_M in the fibered cone (of points projecting to the fibered face) defined by $\psi = (0, 1)$ is the set

$$\Psi = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : b > 0, -b < a < b\}$$

as shown in Figure 3. For the rest of this paper, we will only be concerned with elements of Ψ with connected fibers, that is,

$$\bar{\Psi} = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : b > 0, -b < a < b, \gcd(a, b) = 1\}.$$

The elements of $\bar{\Psi}$ are in one-to-one correspondence with the rational points on the fibered face defined by ψ , which can be thought of as the projectivization of Ψ .

The Alexander polynomial for L is given by

$$\Delta_L(x, u) = u^2 - u(1 - x - x^{-1}) + 1 \quad (3)$$

(see Rolfsen’s table [Rolf76]), and the Teichmuller polynomial is given by

$$\Theta_L(x, u) = u^2 - u(1 + x + x^{-1}) + 1 \quad (4)$$

(see [McM1] 11.I).

Specialization to the element $(a, b) \in H^1(M; \mathbb{Z})$ discussed in Section 2 is the same as plugging (t^a, t^b) into the equations for the Alexander and Teichmuller polynomials.

Proposition 3.1 *If $(a, b) \in \bar{\Psi}$, then the associated monodromy $\phi_{(a,b)}$ is pseudo-Anosov and its homological dilatation is the maximum norm among roots of the polynomial*

$$\Delta_L(t^a, t^b) = t^{2b} - t^b(1 - t^a - t^{-a}) + 1,$$

and the geometric dilatation is the largest real root $\lambda_{(a,b)}$ of

$$\Theta_L(t^a, t^b) = t^{2b} - t^b(1 + t^a + t^{-a}) + 1.$$

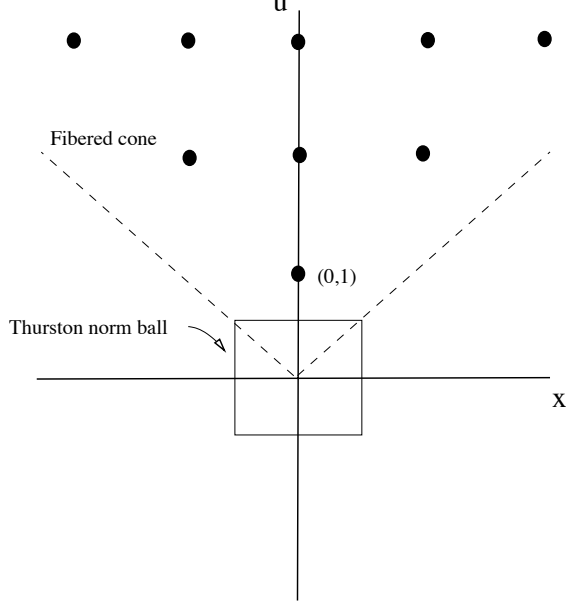


Figure 3: Fibered cone Ψ containing $\psi = (0, 1)$.

Corollary 3.2 *If $(a, b) \in \overline{\Psi}$, then the associated monodromy $\phi_{(a,b)}$ is orientable if a is odd and b is even.*

Proof. If a is odd and b is even, then the roots of $\Theta_L(t^a, t^b)$ are the negatives of the roots of $\Delta_L(t^a, t^b)$. This implies that the geometric and homological dilatations of $\phi_{(a,b)}$ are equal, and therefore $\phi_{(a,b)}$ is orientable. \square

Later in this section, we will show the converse. In preparation, we consider how the monodromy behaves near the boundary of $S_{(a,b)}$ (Corollary 3.8).

Proposition 3.3 *Let $\phi_{(a,b)} : S_{(a,b)} \rightarrow S_{(a,b)}$ be the monodromy associated to (a, b) . The boundary components of $S_{(a,b)}$ consists of $\gcd(3, a)$ components coming from $T(K_1)$ and $\gcd(3, b)$ coming from $T(K_2)$. Thus, the total number of boundary components of $S_{(a,b)}$ is given by*

$$\begin{cases} 2 & \text{if } \gcd(3, ab) = 1 \\ 4 & \text{if } 3 \text{ divides } ab \end{cases}$$

Proof. The number of components in $T(K_i) \cap S_{(a,b)}$ is the index of the image of $\pi_1(T(K_i))$ in \mathbb{Z} under the composition of maps

$$\pi_1(T(K_i)) \rightarrow \pi_1(M) \rightarrow \mathbb{Z}$$

induced by inclusion and $\psi_{(a,b)}$.

For $i = 1, 2$, let ℓ_i be the longitude of K_i that is contractible in $S^3 \setminus K_i$. Then, for $T(K_1)$ we have

$$\psi_{(a,b)}(\mu_1) = a \quad \text{and} \quad \psi_{(a,b)}(\ell_1) = 3\psi_{(a,b)}(\mu_2) = 3b,$$

so the number of boundary components contributed by $T(K_1)$ is

$$\gcd(a, 3b) = \gcd(3, a),$$

since we are assuming that $\gcd(a, b) = 1$. The contribution of $T(K_2)$ is computed similarly. \square

Proposition 3.4 *The genus of $S_{(a,b)}$, for $(a,b) \in \overline{\Psi}$ is given by*

$$\begin{aligned} g(S_{(a,b)}) &= |b| + \left(1 - \frac{\gcd(3,a) + \gcd(3,b)}{2}\right) \\ &= \begin{cases} |b| & \text{if } 3 \text{ does not divide } ab \\ |b| - 1 & \text{if } 3|a \text{ or } 3|b. \end{cases} \end{aligned}$$

Proof. From (2) we have

$$2|b| = \chi_-(S_{(a,b)}) = 2g - 2 + \gcd(3,a) + \gcd(3,b),$$

□

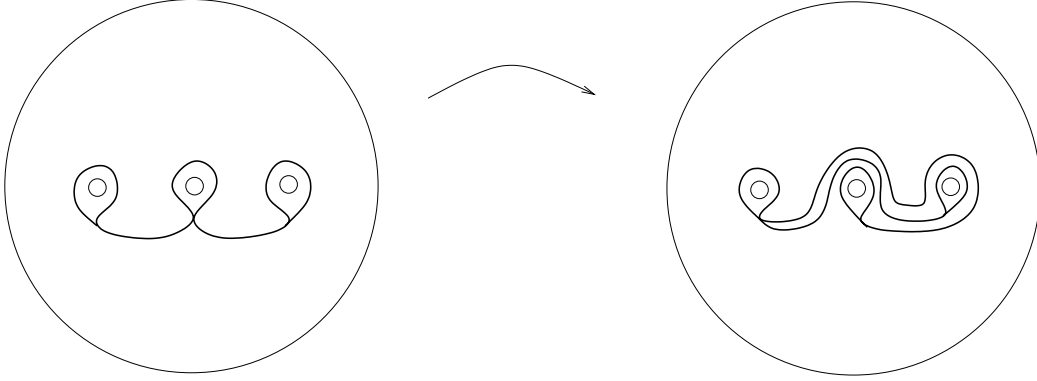


Figure 4: Train track for $\phi : S \rightarrow S$.

Proposition 3.5 *Let $(a,b) \in \overline{\Psi}$, and let \mathcal{F} be a $\phi_{(a,b)}$ -invariant foliation. Then \mathcal{F}*

- (i) *has no interior singularities, and at the m_1 boundary components coming from $T(K_1)$,*
- (ii) *is $(3b/\gcd(3,a))$ -pronged at the $\gcd(3,a)$ boundary components coming from $T(K_1)$, and*
- (iii) *is $(b/\gcd(3,b))$ -pronged at the $\gcd(3,b)$ boundary components coming from $T(K_2)$.*

Proof. Let \mathcal{L} be the lamination of M defined by suspending \mathcal{F} over M considered as the mapping torus of ϕ . From the train track for ϕ (Figure 4), one sees that each of the boundary components of S are one-pronged, and that there are no other singularities. It follows that \mathcal{L} has no singularities outside a neighborhood of the K_i , and near each K_i the leaves of \mathcal{L} come together at a simple closed curve $\gamma_i \in H_1(T(K_i))$. Write

$$\gamma_i = r_i \mu_i + s_i \ell_i$$

for $i = 1, 2$.

For $(a,b) \in \overline{\Psi}$, the number of intersections of γ_i with $S_{(a,b)}$ is the image of γ_i under the epimorphism

$$\psi_{(a,b)} : \pi_1(M) \rightarrow \mathbb{Z}$$

defining the fibration. Figure 4 shows that $s_1 = 1$ and $r_2 = 1$. Using the identities

$$\begin{aligned} s_1 &= 1 & \lambda_1 &= 3\mu_2, \\ r_2 &= 1 & \lambda_2 &= 3\mu_1, \end{aligned}$$

we have

$$\begin{aligned}\psi_{(a,b)}(\gamma_1) &= r_1\psi_n(\mu_1) + 3\psi_n(\mu_2) = r_1a + 3b \\ \psi_{(a,b)}(\gamma_2) &= \psi_n(\mu_2) + 3s_2\psi_n(\mu_1) = 3s_2a + b.\end{aligned}$$

This implies that $\phi_{(a,b)}$ is $(r_1a + 3b)/m_1$ -pronged at m_1 boundary components and $(3s_2a + b)/m_2$ -pronged at m_2 boundary components. We find r_1 and s_2 by looking at some particular examples.

In general, if $f : \Sigma \rightarrow \Sigma$ is pseudo-Anosov on a compact oriented surface Σ with genus g and n_1, \dots, n_k are the number of prongs at the singularities and boundary components, then by the Poincaré-Hopf theorem

$$\sum_{i=1}^k (n_i - 2) = 4g - 4. \quad (5)$$

For $(a, b) = (1, n)$, n not divisible by 3, we have two singularities with number of prongs given by:

$$\begin{aligned}\psi_n(\gamma_1) &= r_1 + 3n \\ \psi_n(\gamma_2) &= 3s_2 + n.\end{aligned}$$

Plugging into (5) gives

$$r_1 + 3s_2 = 0.$$

Let $s = s_2$. The mapping class $\phi_{(1,2)}$ is the unique genus 2 pseudo-Anosov mapping class with dilatation equal to λ_2 [CH08][LT09], and has one 6-pronged singularity (see, for example, [HK06]). Thus, $s = 0$ and we have

$$\gamma_1 = \ell_1 = 3\mu_2$$

and

$$\gamma_2 = \mu_2.$$

The claim follows. □

Corollary 3.6 *The map $\phi_{(a,b)}$ has singularities with number of prongs given by:*

$$\begin{cases} (3b, b) & \text{if } \gcd(3, ab) = 1 \\ (3b, b/3, b/3, b/3) & \text{if } \gcd(3, b) = 3 \\ (b, b, b, b) & \text{if } \gcd(3, a) = 3 \end{cases}$$

Remark 3.7 *The degrees of the singularities is obtained by subtracting two from the number of prongs.*

Corollary 3.8 *If b is odd, then $\phi_{(a,b)}$ is not orientable.*

Proof. By Corollary 3.6, the number of prongs at each boundary component is odd if b is odd. Thus, $\phi_{(a,b)}$ is not locally orientable near the boundary components. □

Corollary 3.9 *For $(a, b) \in \overline{\Psi}$, $\phi_{(a,b)}$ is 1-pronged at one or more boundary components of $S_{(a,b)}$ if and only if $(a, b) \in \{(0, 1), (\pm 1, 3), (\pm 2, 3)\}$.*

Corollary 3.10 *If $(a, b) \notin \{(0, 1), (\pm 1, 3), (\pm 2, 3)\}$, then $\phi_{(a,b)}$ extends to the closure of $S_{(a,b)}$ over the boundary components to a mapping class $\overline{\phi}_{(a,b)}$ with the same dilatation as $\phi_{(a,b)}$.*

Proposition 3.11 *Table 2 below describes the pairs $(a, b) \in \overline{\Psi}$ that give rise to an orientable (or non-orientable) genus g pseudo-Anosov mapping class. (Here $g \geq 4$.)*

$g \pmod{6}$	<i>orientable</i>	<i>non-orientable</i>
0	<i>no example</i>	$b = g + 1, a = 0 \pmod{3}$
1	$b = g + 1, a = 3 \pmod{6}$	$b = g, a = 1, 2 \pmod{3}$
2	$b = g, a = 1, 5 \pmod{6}$	$b = g + 1, a = 1, 2 \pmod{3}$
3	$b = g + 1, a = 3 \pmod{6}$	<i>no example</i>
4	$b = g, a = 1, 5 \pmod{6}$	$b = g + 1, a = 0 \pmod{3}$
5	$b = g + 1, a = 1, 5 \pmod{6}$	$b = g, a = 1, 2 \pmod{3}$

Table 2: Fibrations of M according to genus.

4 Dilatations on the fibered face.

Let $\overline{\Psi}$ be the fibered cone discussed in Section 3. Let

$$\begin{aligned} d_g &= \min\{\delta(\psi) : \psi \in \overline{\Psi}, \text{genus of } \psi \text{ is } g\} \\ d_g^+ &= \min\{\delta(\psi) : \psi \in \overline{\Psi}, \text{genus of } \psi \text{ is } g, \text{the monodromy of } \psi \text{ is orientable}\} \end{aligned}$$

In this section, we finish the proofs of Theorem 1.3 and Theorem 1.4 and their consequences by determining d_g and d_g^+ .

Proposition 4.1 *If $|a| < |a'|$, and $(a, b) \in \Psi$, then*

$$\lambda_{(a,b)} < \lambda_{(a',b)}.$$

Proof. The function $\mathcal{Y}((a, b))$ (see Theorem 2.1) is concave and symmetric with respect to reflection around the b -axis. Thus for fixed b , $\delta(\phi_{(a,b)}) < \delta(\phi_{(a',b)})$ as long as a is closer to the b -axis than a' . \square

Corollary 4.2

$$\lim_{g \rightarrow \infty} (d_g)^g = \lim_{g \rightarrow \infty} (d_g^+)^g = \frac{3 + \sqrt{5}}{2}.$$

Proposition 4.3 *For $b = 3$,*

$$\lambda_{(1,3)} = \lambda_{(3,4)}$$

and for any $b \geq 4$,

$$\lambda_{(1,b)} > \lambda_{(3,b+1)}.$$

Proof. Let $\lambda = \lambda_{(3,b+1)}$. Again, we would like to show that $LT_{(1,b)}(\lambda) < 0$. Consider the function

$$f(x) = -x^{b+4} + x^{b+2} + x^6 - x^2 - x + 1.$$

Then for $x > 1$,

$$\begin{aligned} f'(x) &= -(b+4)x^{b+3} + (b+2)x^{b+1} + 6x^5 - 2x \\ &= x[-(b+4)x^{b+2} + (b+2)x^b + 6x^4 - 2] \\ &< x[-(b+4)x^{b+2} + (b+3)x^b] \\ &< 0. \end{aligned}$$

Since $f(1) = 0$, this implies that $f(x) < 0$, for $x > 1$. Using this fact, we see that

$$\begin{aligned}
LT_{(1,b)}(\lambda) &= LT_{(1,b)}(\lambda) - LT_{(3,b+1)}(\lambda) \\
&= \lambda^{2b} - \lambda^{2b+2} + (\lambda^{b+4} - \lambda^{b+1}) + (\lambda^{b-2} - \lambda^{b-1}) + (\lambda^{b+1} - \lambda^b) \\
&= -\lambda^{2b+2} + \lambda^{2b} + \lambda^{b+4} - \lambda^b - \lambda^{b-1} + \lambda^{b-2} \\
&= \lambda^{b-2}[-\lambda^{b+4} + \lambda^{b+2} + \lambda^6 - \lambda^2 - \lambda + 1] \\
&< 0.
\end{aligned}$$

□

Remark 4.4 Since $\lambda_{(1,3)} = \lambda_{(3,4)}$, it follows that the dilatation δ_3^+ is realized by both an orientable and a non-orientable mapping class in Ψ .

Lemma 4.5 For $n \geq 2$, Then

$$\lim_{n \rightarrow \infty} (\lambda_{(a,n)})^n = \frac{3 + \sqrt{5}}{2},$$

for any fixed a .

Proof. The projections of the lattice points $(a, n) \in \Lambda_M$ on the fibered face of ψ converge to $(0, 1/2)$.

□

Putting together Proposition 4.1 and Proposition 4.3, we have the following.

Proposition 4.6 The sequences $\lambda_{(1,b)}$ and $\lambda_{(3,b)}$ satisfy:

$$\lambda_{(1,b)} > \lambda_{(3,b+1)} > \lambda_{(1,b+1)}.$$

Proposition 4.7 The following table describes the pairs $(a, b) \in \bar{\Psi}$ that give rise to minimal dilatation examples for each genus g . Here unconstrained means not constrained to be orientable.

$g \bmod 6$	constrained	unconstrained
0	no example	$(3, g + 1)$
1	$(3, g + 1)$	$(3, g + 1)$
2	$(1, g)$	$(1, g + 1)$
3	$(3, g + 1)$	$(3, g + 1)$
4	$(1, g)$	$(3, g + 1)$
5	$(1, g + 1)$	$(1, g + 1)$

Table 3: Pairs (a, b) giving smallest dilatations.

Proposition 4.7 and Corollary 3.10 complete the proofs of Theorem 1.3 and Theorem 1.4. A pictorial view of the elements of Ψ giving the least dilatations for each genus up to 12 is given in Figure 5.

Remark 4.8 For $g = 5$, δ_5^+ is realized by a mapping class associated to the E_{10} Coxeter diagram (see [Lein04]). It is also the monodromy of the $(-2, 3, 7)$ -pretzel link [Hir04]. One can verify directly that the monodromy $\phi_{1,6}$ is the same as this mapping class by noting that the $(-2, 3, 7)$ -pretzel link can be obtained from the braid $\sigma_1\sigma_2^{-1}$ by composing the braid with two full twists and taking the braid closure.

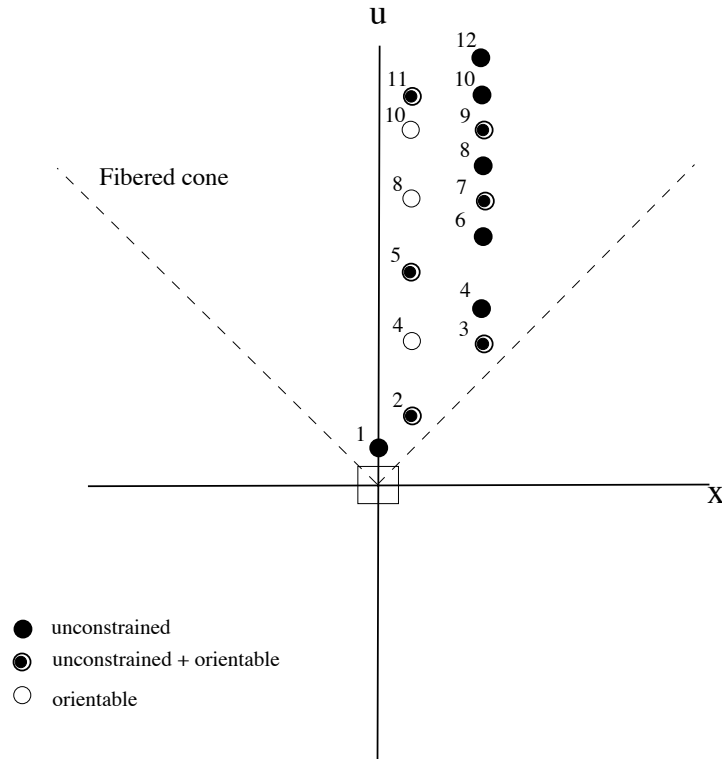


Figure 5: Minima for d and d^+ in genus $g = 1, \dots, 12$.

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