SMALL DILATATION MAPPING CLASSES COMING FROM THE SIMPLEST HYPERBOLIC BRAID

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Abstract. In this paper we study the small dilatation pseudo-Anosov mapping classes arising from fibrations over the circle of a single 3-manifold, the mapping torus for the "simplest hyperbolic braid". The dilatations that occur include the minimum dilatations for orientable pseudo-Anosov mapping classes for genus $g = 2, 3, 4, 5$ and 8. We obtain the "Lehmer example" in genus $g = 5$, and Lanneau and Thiffeault's conjectural minima in the orientable case for all genus g satisfying $g = 2$ or $4 (\mathrm{mod}\ 6)$. Our examples show that the minimum dilatation for orientable mapping classes is strictly greater than the minimum dilatation for non-orientable ones when $g = 4, 6$ or 8. We also prove that if δ_g is the minimum dilatation of pseudo-Anosov mapping classes on a genus g surface, then

$$\limsup_{g \to \infty} (\delta_g)^g \leq \frac{3+\sqrt{5}}{2}.$$

1. Introduction

Let S_g be a closed oriented surface of genus $g \geq 1$, and let Mod_g be the *mapping class group* of isotopy classes of orientation preserving self-homeomorphisms of S_g. A mapping class $\phi \in \mathrm{Mod}_g$ is called *pseudo-Anosov* if S_g admits a pair of ϕ-invariant, transverse measured, singular foliations on which ϕ acts by stretching transverse to one foliation by a constant $\lambda(\phi) > 1$ and contracting transverse to the other by $\lambda(\phi)^{-1}$. The constant $\lambda(\phi)$ is called the *(geometric) dilatation* of ϕ. A mapping class is pseudo-Anosov if it is neither periodic nor reducible [26, 7, 3]. Denote by $\mathrm{Mod}_g^{\mathrm{pA}}$ the set of pseudo-Anosov mapping classes in Mod_g.

A pseudo-Anosov mapping class ϕ is defined to be *orientable* if its invariant foliations are orientable. We will denote the set of orientable pseudo-Anosov mapping classes by $\mathrm{Mod}_g^{\mathrm{pA}+}$. Let $\lambda_{\mathrm{hom}}(\phi)$ be the spectral radius of the action of ϕ on the first homology of S. Then

$$\lambda_{\mathrm{hom}}(\phi) \leq \lambda(\phi),$$

with equality if and only if ϕ is orientable (see, for example, [16, 15]).

The dilatations $\lambda(\phi)$ satisfy reciprocal monic integer polynomials of degree bounded from above by $6g - 6$ [26]. If ϕ is orientable the degree is bounded by $2g$. For fixed g, it follows that $\lambda(\phi)$ achieves a minimum $\delta_g > 1$ on $\mathrm{Mod}_g^{\mathrm{pA}}$ (see also, [2, 12]). Let

$$\mathrm{Mod}_g^{\mathrm{pA}+} \subset \mathrm{Mod}_g^{\mathrm{pA}}$$

be the subset of orientable pseudo-Anoosv mapping classes, and let δ_g^+ be the minimum dilatation among elements of $\mathrm{Mod}_g^{\mathrm{pA}+}$.

In this paper, we address the following question (cf. [23, 20, 5]):

Question 1.1. *What is the behavior of δ_g and δ_g^+ as functions of g?*

So far, exact values of δ_g have only been found for $g \leq 2$. For $g = 1$, the derivative map determines an identification $\mathrm{Mod}_1 = \mathrm{SL}(2;\mathbb{Z})$, and

$$\delta_1 = \frac{3+\sqrt{5}}{2}.$$

For a monic integer polynomial $p(x)$, the *house* of $p(x)$, written $|p|$, is the absolute value of the largest root of p. For $g = 2$, Cho and Ham [4] show that δ_2 is given by

$$|t^4 - t^3 - t^2 - t + 1| \approx 1.72208.$$

In the orientable case more is known due to recent results of Lanneau and Thiffeault [16]. Given $(a, b) \in \mathbb{Z} \oplus \mathbb{Z}$ with $0 < a < b$, let

$$LT_{(a,b)}(t) = t^{2b} - t^b(1 + t^a + t^{-a}) + 1,$$

and let

$$\lambda_{(a,b)} = |LT_{(a,b)}(t)|.$$

Theorem 1.2 (Lanneau-Thiffeault [16] Theorems 1.2 and 1.3)**.** *For $g = 2, 3, 4, 6,$ and 8,*

$$\lambda_{(1,g)} \leq \delta_g^+$$

with equality when $g = 2, 3$ or 4.

For $g = 2$, the value of δ_2^+ was first determined by Zhirov [27]. For $g = 5$, Lanneau and Thiffeault show that δ_5^+ equals Lehmer's number (≈ 1.17628) [17]. This dilatation is realized as a product of multi-twists along a curve arrangement dual to the E_{10} Coxeter graph (see [18, 10]), and as the monodromy of the (-2,3,7)-pretzel knot (see [9]). Lanneau and Thiffeault also find a lower bound for δ_7^+. An example realizing this bound can be found in [1] (p.4) and [14] (Theorem 1.12).

Based on their results, Lanneau and Thiffeault ask:

Question 1.3 ([16] Question 6.1)**.** *Is $\delta_g^+ = \lambda_{(1,g)}$ for all even g?*

For convenience, we will call the affirmative answer to their question the *LT-conjecture*.

In our first result, we improve on the following previous best bounds for the minimum dilatation of infinite families

$$(\delta_g)^g \leq (\delta_g^+)^g \leq 2 + \sqrt{3}$$

found in [22, 11].

Theorem 1.4. *If $g = 0, 1, 3$ or $4 (\mathrm{mod}\ 6)$, $g \geq 3$, then*

$$\delta_g \leq \lambda_{(3,g+1)},$$

and if $g = 2$ or $5 (\mathrm{mod}\ 6)$ and $g \geq 5$, then

$$\delta_g \leq \lambda_{(1,g+1)}.$$

g	orientable	degrees of singularities	unconstrained	degrees of singularities
1	2.61803*	no sing.	2.61803*	no sing.
2	1.72208*	(4)	1.72208*	(4)
3	1.40127*	$(2, 2, 2, 2)$	1.40127	(2,2,2,2)
4	1.28064*	(10,2)	1.26123	(3,3,3,3)
5	1.17628*	(16)	1.17628	(16)
6	-	-	1.1617	(5,5,5,5)
7	1.13694	(6,6,6,6)	1.13694	(6,6,6,6)
8	1.12876*	(22,6)	1.1135	(25,1,1,1)
9	1.1054	(8.8.8.8)	1.1054	(8,8,8,8)
10	1.10149	(28,8)	1.09466	(9,9,9,9)
11	1.08377	(34,2,2,2)	1.08377	(34,2,2,2)
12	-	-	1.07874	(11,11,11,11)

TABLE 1. Minimal orientable and unconstrained dilatations coming from M_{sb}

For the orientable case, our results complement those of Lanneau and Thiffeault for $g = 2$, or 4(mod 6).

Theorem 1.5. *Let $g \geq 3$. Then*

$$\begin{array}{ll} \delta_g^+ \leq \lambda_{(3,g+1)} & \text{if } g = 1 \text{ or } 3 (\mathrm{mod}\ 6), \\ \delta_g^+ \leq \lambda_{(1,g)} & \text{if } g = 2 \text{ or } 4 (\mathrm{mod}\ 6), \text{ and} \\ \delta_g^+ \leq \lambda_{(1,g+1)} & \text{if } g = 5 (\mathrm{mod}\ 6). \end{array}$$

Putting Theorem 1.5 together with Lanneau and Thiffeault's lower bound for $g = 8$ gives:

Corollary 1.6. *The minimal dilatation for orientable pseudo-Anosov mapping classes for genus 8 is given by*

$$\delta_8^+ = \lambda_{(1,8)}.$$

The following is a table of the minimal dilatations that arise in this paper's examples for genus 1 through 12. All numbers in the table are truncated to 5 decimal places. An asterisk $*$ marks the numbers that have been verified to equal δ_g^+ (resp., δ_g). For singularity-type, we use the convention that $(a_1, \ldots, a_k)$ means that the singularities of the invariant foliations have degrees $a_1, \ldots, a_k$ (see Lanneau and Thiffeault's notation [16] p.3). The singularity-types for our examples are derived from the formula given in Corollary 3.6.

For $g = 2, 3, 4$, and 5, our orientable examples agree both in dilatation and in singularity-type with the previously known minimizing examples (see [16] §3, §4, §6). For $g = 8$, our example agrees with the singularity-type anticipated by Lanneau and Thiffeault [16] (6.4). We prove that the known minimial dilatation examples for $g = 2, 3, 4, 5$, and 8 arise as the monodromy of fibrations of a single 3-manifold M_{sb}. For $g = 7$, our minimal example gives a larger dilatation than δ_7^+. (The dilatation δ_7^+ is realized in [14] and [1].)

Lanneau and Thiffeault show that $\delta_5^+ \leq \delta_6^+$, and hence δ_g^+ is not strictly monotone decreasing (cf. [5] Question 7.2). Theorem 1.5 implies the following stronger statement.

Proposition 1.7. *If the LT-conjecture is true, then $\delta_g^+ \leq \delta_{g+1}^+$, whenever $g = 5(\mathrm{mod}\ 6)$.*

Another consequence concerns the question of when the inequality $\delta_g \leq \delta_g^+$ is strict. In [14] and [1] it is shown that $\delta_5 < \delta_5^+$. Table 1 shows the following.

Corollary 1.8. *For $g = 4, 6,$ and 8 we have*

$$\delta_g < \delta_g^+.$$

Theorem 1.4 and Proposition 4.3 imply the following.

Proposition 1.9. *If the LT-conjecture is true, then for all even $g \geq 4$ we have*

$$\delta_g < \delta_g^+.$$

For large g, it is known that δ_g and δ_g^+ converges to 1. Furthermore,

$$\log(\delta_g) \asymp \frac{1}{g} \qquad \text{and} \qquad \log(\delta_g^+) \asymp \frac{1}{g} \tag{1}$$

(see [23, 20, 22, 11]). The LT-conjecture together with (1) leads to the natural question:

Question 1.10 (cf. [20] p.551, [5] Problem 7.1)**.** *Do the sequences*

$$(\delta_g)^g \qquad \text{and} \qquad (\delta_g^+)^g$$

converge as g grows? What is the limit?

Theorem 1.4 and Theorem 1.5 imply the following.

Theorem 1.11.

$$\limsup_{g\to\infty} (\delta_g)^g \leq \frac{3+\sqrt{5}}{2}$$

and

$$\limsup_{g\neq 0 (\mathrm{mod} 6)} (\delta_g^+)^g \leq \frac{3+\sqrt{5}}{2}.$$

This leads to the question:

Question 1.12 (Golden Mean Question)**.** *Do the sequences $(\delta_g)^g$ and $(\delta_g^+)^g$ satisfy*

$$\lim_{g\to\infty}(\delta_g)^g = \lim_{g\to\infty}(\delta_g^+)^g = \frac{3+\sqrt{5}}{2} = (\textit{golden mean})^2 \ ?$$

For any pseudo-Anosov mapping class ϕ, let $M(\phi)$ be the mapping torus of ϕ. Conversely, given a compact hyperbolic 3-manifold with torus boundary components M, let $\Phi(M)$ be the collection of pseudo-Anosov mapping classes ϕ such that $M = M(\phi)$. Let Σ be the suspensions of singularities of the stable and unstable foliations of ϕ and let

$$M^*(\phi) = M(\phi) \setminus \Sigma.$$

Theorem 1.13 ([6] Theorem 1.1)**.** *The set*

$$\mathcal{T}_P = \{M^*(\phi) \ : \ \phi \in Mod_g^{pA}, \lambda(\phi) \leq P^{\frac{1}{g}}\}$$

is finite for any $P > 1$.

The asymptotic equations (1) and Theorem 1.13 imply that

$$\mathcal{T} = \{M^*(\phi) \ : \ \phi \in \mathrm{Mod}_g^{\mathrm{pA}}, \lambda(\phi) = \delta_g\}$$

and

$$\mathcal{T}^+ = \{M^*(\phi) \ : \ \phi \in \mathrm{Mod}_g^{\mathrm{pA}+} \lambda(\phi) = \delta_g^+\}$$

are finite.

This leads to the question:

Question 1.14. *How large are the sets $\mathcal{T}$ and $\mathcal{T}^+$?*

If the LT-conjecture is true, then our results imply that a single 3-manifold M_{sb} would realize δ_g^+ for all $g = 2, 4 (\mathrm{mod}\ 6)$. The manifold M_{sb} is the complement of the 6_2^2 braid (see Rolfsen's tables [24], and Figure 1). Another 3-manifold that produces small dilatation mapping classes is the complement $M_{\text{-}2,3,8}$ of the $(-2, 3, 8)$-pretzel link in S^3. These have been studied independently by Kin and Takasawa [14] and Aaber and Dunfield [1]. For certain genera the mapping classes in $\Phi(M_{\text{-}2,3,8})$ have smaller dilatation than the minima realized by M_{sb}, but the asymptotic behavior of the minimal dilatations for large genus, supports the affirmative to Question 1.12. Both $M_{-2,3,8}$ and M_{sb} can be obtained from the *magic manifold* by Dehn fillings [19]. The pseudo-Anosov braid monodromies with smallest known dilatations found in [11] are also realized on the magic manifold [13].

Section 2 contains a brief review of Thurston norms, fibered faces and the Teichmüller polynomial. These are the basic tools used in this paper. In Section 3 we describe our family of examples, and in Section 4 we prove Theorem 1.4 and Theorem 1.5.

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2. Background and tools

In this section we give a brief review of invariants and properties of fibrations of a hyperbolic 3-manifold M, emphasizing the tools that we will use in the rest of the paper. For more details see, for example, [25, 7, 20, 21].

The theory of fibered faces of the Thurston norm ball and the existence of Teichmüller polynomials provides a way to study in a single picture a collection of pseudo-Anosov mapping classes defined on surfaces of different Euler characteristics and genera. Assume M is a compact hyperbolic 3-manifold with boundary. Given an embedded orientable surface S on M, let $\chi_-(S)$ be the sum of $|\chi(S_i)|$, where S_i are the connected components of S with negative Euler characteristic. The Thurston norm of $\psi \in \mathrm{H}^1(M;\mathbb{Z})$ is defined to be

$$||\psi||_T = \min \chi_-(S),$$

where the minimum is taken over oriented embedded surfaces $(S, \partial S) \subset (M, \partial M)$ such that the class of $(S, \partial S)$ in $\mathrm{H}_2(M, \partial M; \mathbb{Z})$ is the Poincaré dual of ψ.

Elements of $\mathrm{H}^1(M;\mathbb{Z})$ are canonically associated with epimorphisms

$$\pi_1(M;\mathbb{Z}) \to \mathbb{Z}.$$

We thus make the following natural identification:

$$\mathrm{H}^1(M;\mathbb{Z}) = \mathrm{Hom}(\pi_1(M), \mathbb{Z}) = \mathrm{Hom}(\mathrm{H}_1(M;\mathbb{Z}), \mathbb{Z}).$$

We consider this as a lattice Λ_M inside $\mathbb{R}^{b_1(M)}$, where $b_1(M)$ is the first Betti number of M. If $\psi \in \Lambda_M$ corresponds to a fibration

$$\psi : M \to S^1$$

we say that ψ is *fibered*. In this case the Thurston norm of ψ is given by

$$||\psi||_T = \chi_-(S),$$

where S is homeomorphic to the fiber of ψ. Let

$$\Psi(M) = \{\psi : M \to S^1 \ : \ \psi \text{ is a fibration}\}.$$

The *monodromy* ϕ of $\psi \in \Psi(M)$ is the mapping class $\phi : S \to S$, such that M is the mapping torus of ϕ, and ψ is the natural projection to S^1. Since M is hyperbolic, ϕ is pseudo-Anosov.

Let B be the unit ball in $\mathbb{R}^{b_1(M)}$ with respect to the extended Thurston norm.

Theorem 2.1 ([25])**.** *The Thurston norm ball B is a convex polyhedron and for any top-dimensional open face F of B, $(F \cdot \mathbb{R}^+) \cap \Psi(M)$ is either empty or equal to $(F \cdot \mathbb{R}^+) \cap \Lambda_M$.*

If $(F \cdot \mathbb{R}^+) \cap \Psi(M) \neq \emptyset$, we say F is a *fibered face* of B. An element of $\Psi(M)$ is called *primitive* if its fiber is connected. The elements of Λ_M project to the rational points on the boundary of B. If F is a fibered face, then each rational point x on F corresponds to a unique primitive element $\psi_x \in \Psi(M)$, namely the element of $(x \cdot \mathbb{R}^+) \cap \Psi(M)$ that lies closest to the origin.

Theorem 2.2 ([8], Theorem E)**.** *There is a continuous function $\mathcal{Y}$, homogeneous of degree one, defined on the fibered cone in $\mathbb{R}^{b_1(M)}$, so that if ψ is fibered with monodromy ϕ_ψ, then*

$$\mathcal{Y}(\psi) = \frac{1}{\log(\lambda(\phi_\psi))}.$$

The function $\mathcal{Y}$ is concave and tends to zero along the boundary of the cone.

Corollary 2.3. *For each fibered face F,*

$$\overline{\lambda}(\psi) = \lambda(\phi_\psi)^{||\psi||_T},$$

extends to a continuous function on $F \cdot \mathbb{R}^+$ that is constant on rays through the origin, and $\overline{\lambda}$ achieves a unique minimum on F.

Let G be a group and $\psi : G \to \mathbb{Z}$ a homomorphism. If $f \in \mathbb{Z}[G]$ is given by

$$f = \sum_{g \in G} \alpha_g g,$$

then the *specialization* of f at ψ is the polynomial in $\mathbb{Z}[t]$ defined by

$$f^{\psi}(t) = \sum_{g \in G} \alpha_g t^{\psi(g)}.$$

Theorem 2.4 ([20]). *Let F be a fibered face for a 3-manifold M, and let $G = H_1(M;\mathbb{Z})$. Then there is an element $\theta_F \in \mathbb{Z}[G]$ such that for all integral lattice points ψ in the fibered cone of F,*

$$\lambda(\phi_\psi) = |\theta_F^{\psi}|.$$

The polynomial θ_F is called the *Teichmüller polynomial* of M for the fibered face F.

3. The mapping torus for the simplest hyperbolic braid

We now look at a particular 3-manifold, and study properties of its fibrations. This example has also been studied in ([20] §11), and the first part of this section will be a review of what is found there.

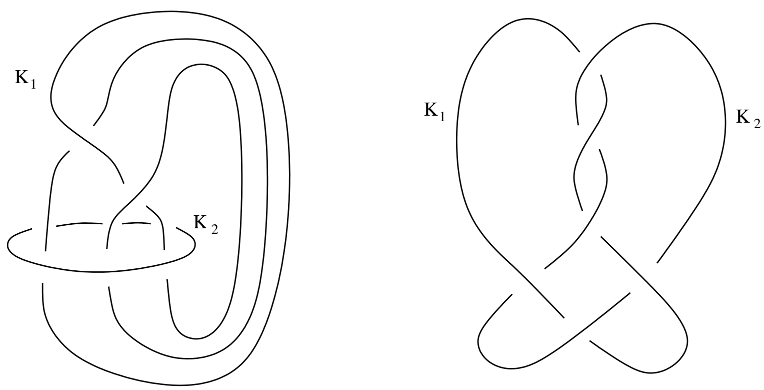

Figure 1. Two diagrams for the link 6_2^2.

Let $M = S^3 \setminus N(L)$, where L is the link drawn in two ways in Figure 1, and $N(L)$ is a tubular neighborhood. As seen from the left diagram in Figure 1, M fibers over the circle with fiber a sphere with four boundary components $S_{0,4}$. Let $\psi_0 : M \to S^1$ be the corresponding fibration, and let $\phi_0 : S_{0,4} \to S_{0,4}$ be the monodromy. Then ϕ_0 is the mapping class associated to the braid written with respect to standard generators as $\sigma_1\sigma_2^{-1}$ (see Figure 2) and its dilatation is given by

$$\lambda(\phi_0) = \frac{3+\sqrt{5}}{2}.$$

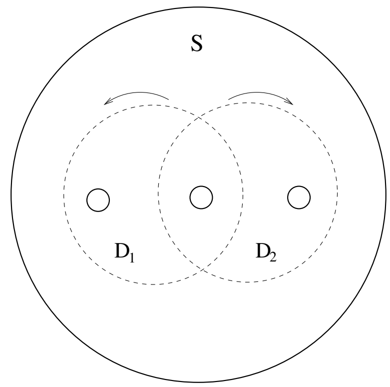

FIGURE 2. Braid monodromy associated to $\sigma_1\sigma_2^{-1}$.

The braid $\sigma_1\sigma_2^{-1}$ has been called the "simplest hyperbolic braid" (cf. [20] §11).

Let K_1 and K_2 be the components of L as drawn in Figure 1. Let μ_1 be the meridian of K_1 and μ_2 be the meridian of K_2. These determine coordinate functions for $H^1(M;\mathbb{Z})$

$$(\mu_1, \mu_2)(\psi) = (\psi(\mu_1), \psi(\mu_2)) \in \mathbb{Z} \times \mathbb{Z}.$$

With respect to these coordinates, the Thurston norm and the Alexander norm both are given by

$$||(a,b)|| = \max\{2|a|, 2|b|\}. \tag{2}$$

The lattice points Λ_M in the fibered cone $F \cdot \mathbb{R}^+$ defined by $\psi = (0,1)$ is the set

$$\Psi = \{(a,b) \in \mathbb{Z} \times \mathbb{Z} \ : \ b > 0, -b < a < b\}$$

as shown in Figure 3. For the rest of this paper, we will only be concerned with the subset $\Psi_{\text{prim}} \subset \Psi$ consisting of elements of Ψ with connected fibers, i.e., the *primitive elements*. Thus,

$$\Psi_{\text{prim}} = \{(a,b) \in \mathbb{Z} \times \mathbb{Z} \ : \ b > 0, -b < a < b, \gcd(a,b) = 1\}.$$

The Alexander polynomial for L is given by

$$\Delta_L(x,u) = u^2 - u(1 - x - x^{-1}) + 1 \tag{3}$$

(see Rolfsen's table [24]), and the Teichmüller polynomial is given by

$$\Theta_L(x,u) = u^2 - u(1 + x + x^{-1}) + 1 \tag{4}$$

(see [20] p.47).

Specialization to the element $(a,b) \in \mathrm{H}^1(M;\mathbb{Z})$ is the same as plugging (t^a, t^b) into the equations for the Alexander and Teichmüller polynomials (see Section 2).

Proposition 3.1. *If $(a,b) \in \Psi_{prim}$, then the associated monodromy $\phi_{(a,b)}$ is pseudo-Anosov with geometric dilatation given by*

$$\lambda_{(a,b)} = |\Theta_L(t^a, t^b)| = |t^{2b} - t^b(1 + t^a + t^{-a}) + 1|,$$

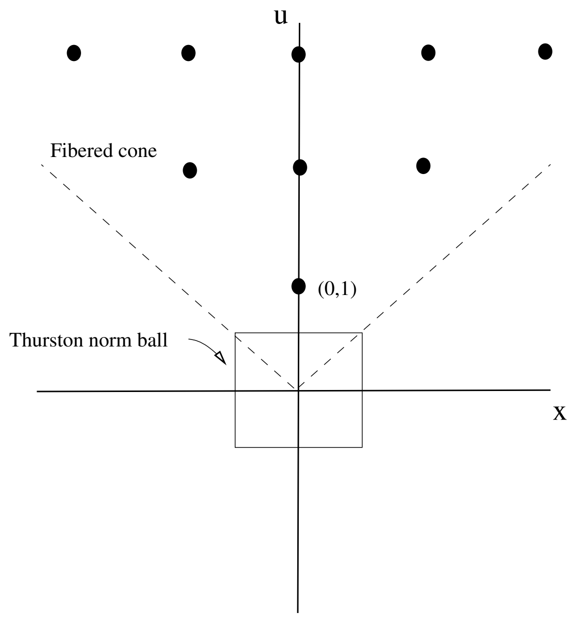

FIGURE 3. Fibered cone Ψ containing $\psi = (0,1)$.

and homological dilatation given by

$$\lambda_{(a,b)}^{hom} = |\Delta_L(t^a, t^b)| = |t^{2b} - t^b(1 - t^a - t^{-a}) + 1|.$$

Corollary 3.2. *If $(a,b) \in \Psi_{prim}$, then the associated monodromy $\phi_{(a,b)}$ is orientable if a is odd and b is even.*

Proof. If a is odd and b is even, then the roots of $\Theta_L(t^a, t^b)$ are the negatives of the roots of $\Delta_L(t^a, t^b)$. This implies that the geometric and homological dilatations of $\phi_{(a,b)}$ are equal, and therefore $\phi_{(a,b)}$ is orientable. $\square$

Later in this section, we prove the converse of Corollary 3.2. First we consider how the monodromy behaves near the boundary of $S_{(a,b)}$.

Proposition 3.3. *Let $\phi_{(a,b)} : S_{(a,b)} \to S_{(a,b)}$ be the monodromy associated to $(a,b) \in \Psi_{prim}$. The boundary components of $S_{(a,b)}$ has $\gcd(3,a)$ components coming from $T(K_1)$ and $\gcd(3,b)$ coming from $T(K_2)$. Thus, the total number of boundary components of $S_{(a,b)}$ is given by*

$$\begin{cases} 2 & \textit{if } \gcd(3,ab) = 1 \\ 4 & \textit{if } \gcd(3,ab) = 3 \end{cases}$$

Proof. The number of components in $T(K_i) \cap S_{(a,b)}$ is the index of the image of $\pi_1(T(K_i))$ in $\mathbb{Z}$ under the composition of maps

$$\pi_1(T(K_i)) \to \pi_1(M) \to \mathbb{Z}$$

induced by inclusion and $\psi_{(a,b)}$.

For $i = 1, 2$, let ℓ_i be the longitude of K_i that is contractible in $S^3 \setminus K_i$. Then, for $T(K_1)$ we have

$$\psi_{(a,b)}(\mu_1) = a \quad \text{and} \quad \psi_{(a,b)}(\ell_1) = 3\psi_{(a,b)}(\mu_2) = 3b,$$

so the number of boundary components contributed by $T(K_1)$ is

$$\gcd(a, 3b) = \gcd(3, a),$$

since we are assuming that $\gcd(a, b) = 1$. The contribution of $T(K_2)$ is computed similarly. □

Proposition 3.4. *The genus of $S_{(a,b)}$, for $(a, b) \in \Psi_{prim}$ is given by*

$$\begin{aligned} g(S_{(a,b)}) &= |b| + \left(1 - \frac{\gcd(3, a) + \gcd(3, b)}{2}\right) \\ &= \begin{cases} |b| & \text{if } \gcd(3, ab) = 1 \\ |b| - 1 & \text{if } \gcd(3, ab) = 3. \end{cases} \end{aligned}$$

Proof. Equation (2) gives

$$2|b| = \chi_-(S_{(a,b)}) = 2g - 2 + \gcd(3, a) + \gcd(3, b).$$

□

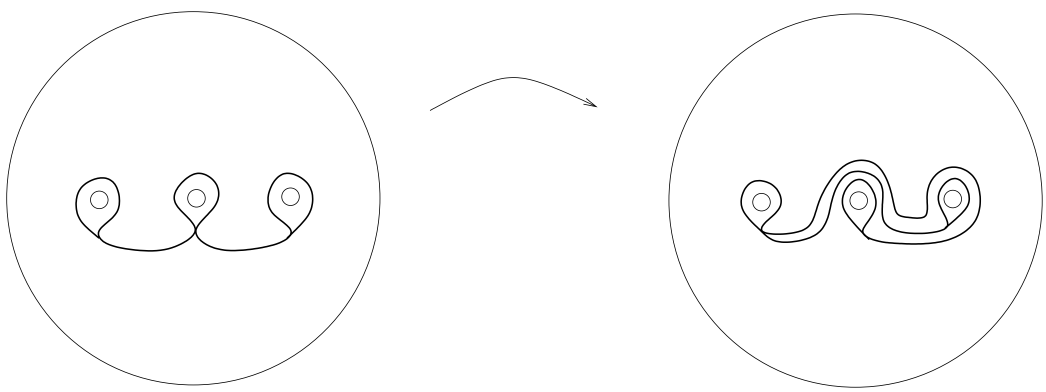

FIGURE 4. Train track for $\phi : S \to S$.

Proposition 3.5. *Let $(a, b) \in \Psi_{prim}$, and let $\mathcal{F}$ be a $\phi_{(a,b)}$-invariant foliation. Then $\mathcal{F}$*

(1) *has no interior singularities,*
(2) *is $(3b/\gcd(3, a))$-pronged at each of the $\gcd(3, a)$ boundary components coming from $T(K_1)$, and*
(3) *is $(b/\gcd(3, b))$-pronged at each of the $\gcd(3, b)$ boundary components coming from $T(K_2)$.*

Proof. Let $\mathcal{L}$ be the lamination of M defined by suspending $\mathcal{F}$ over M considered as the mapping torus of ϕ. From the train track for ϕ (Figure 4), one sees that each of the boundary components of S are one-pronged, and that there are no other singularities. It follows that $\mathcal{L}$ has no singularities outside a neighborhood of the K_i, and near each K_i the leaves of $\mathcal{L}$ come together at a simple closed curve $\gamma_i \in \mathrm{H}_1(T(K_i))$. Write

$$\gamma_i = r_i\mu_i + s_i\ell_i$$

for $i = 1, 2$.

For $(a, b) \in \Psi_{\mathrm{prim}}$, the number of intersections of γ_i with $S_{(a,b)}$ is the image of γ_i under the epimorphism

$$\psi_{(a,b)} : \pi_1(M) \to \mathbb{Z}$$

defining the fibration. Figure 4 shows that $s_1 = 1$ and $r_2 = 1$. Using the identities

$$\begin{aligned} s_1 &= 1 \qquad & \ell_1 &= 3\mu_2, \\ r_2 &= 1 \qquad & \ell_2 &= 3\mu_1, \end{aligned}$$

we have

$$\begin{aligned} \psi_{(a,b)}(\gamma_1) &= r_1\psi_n(\mu_1) + 3\psi_n(\mu_2) = r_1 a + 3b \\ \psi_{(a,b)}(\gamma_2) &= \psi_n(\mu_2) + 3s_2\psi_n(\mu_1) = 3s_2 a + b. \end{aligned}$$

Let $m_1 = \gcd(3, a)$ and $m_2 = \gcd(3, b)$. Then $\phi_{(a,b)}$ is $(r_1 a + 3b)/m_1$-pronged at m_1 boundary components and $(3s_2 a + b)/m_2$-pronged at m_2 boundary components. We find r_1 and s_2 by looking at some particular examples.

In general, if $f : \Sigma \to \Sigma$ is pseudo-Anosov on a compact oriented surface Σ with genus g and $n_1, \dots, n_k$ are the number of prongs at the singularities and boundary components, then by the Poincaré-Hopf theorem

$$\sum_{i=1}^{k}(n_i - 2) = 4g - 4. \tag{5}$$

For $(a, b) = (1, n)$, n not divisible by 3, we have two singularities with number of prongs given by:

$$\begin{aligned} \psi_n(\gamma_1) &= r_1 + 3n \\ \psi_n(\gamma_2) &= 3s_2 + n. \end{aligned}$$

Plugging into (5) gives

$$r_1 + 3s_2 = 0.$$

The mapping class $\phi_{(1,2)}$ is the unique genus 2 pseudo-Anosov mapping class with dilatation equal to λ_2 [4, 16], and has one 6-pronged singularity [11]. Thus, $r_1 = s_2 = 0$ and

$$\gamma_1 = \ell_1 = 3\mu_2$$

and

$$\gamma_2 = \mu_2.$$

The claim follows. □

g (mod 6)	orientable	non-orientable
0	no example	$b = g + 1,\ a = 0 (\mathrm{mod}\ 3)$
1	$b = g + 1,\ a = 3 (\mathrm{mod}\ 6)$	$b = g,\ a = 1, 2 (\mathrm{mod}\ 3)$
2	$b = g,\ a = 1, 5 (\mathrm{mod}\ 6)$	$b = g + 1,\ a = 1, 2 (\mathrm{mod}\ 3)$
3	$b = g + 1,\ a = 3 (\mathrm{mod}\ 6)$	no example
4	$b = g,\ a = 1, 5 (\mathrm{mod}\ 6)$	$b = g + 1,\ a = 0 (\mathrm{mod}\ 3)$
5	$b = g + 1,\ a = 1, 5 (\mathrm{mod}\ 6)$	$b = g,\ a = 1, 2 (\mathrm{mod}\ 3)$

TABLE 2. Fibrations of M according to genus.

Corollary 3.6. *The map $\phi_{(a,b)}$ has singularities with number of prongs (or prong-type) given by*

$$\begin{cases} (3b, b) & \textit{if } \gcd(3, ab) = 1 \\ (3b, b/3, b/3, b/3) & \textit{if } \gcd(3, b) = 3 \\ (b, b, b, b) & \textit{if } \gcd(3, a) = 3 \end{cases}$$

The degree of a singularity and the number of prongs differ by 2, yielding Table 1.

Corollary 3.7. *If b is odd, then $\phi_{(a,b)}$ is not orientable.*

Corollary 3.8. *For $(a, b) \in \Psi_{prim}$, $\phi_{(a,b)}$ is 1-pronged at one or more boundary components of $S_{(a,b)}$ if and only if $(a, b) \in \{(0, 1), (\pm 1, 3), (\pm 2, 3)\}$.*

Corollary 3.9. *If $(a, b) \notin \{(0, 1), (\pm 1, 3), (\pm 2, 3)\}$, then $\phi_{(a,b)}$ extends to the closure of $S_{(a,b)}$ over the boundary components to a mapping class $\overline{\phi}_{(a,b)}$ with the same dilatation as $\phi_{(a,b)}$.*

Table 2 describes the pairs $(a, b) \in \Psi_{\mathrm{prim}}$ that give rise to an orientable (or non-orientable) genus g pseudo-Anosov mapping class. (Here $g \geq 4$.)

4. Minimal dilatations for the fibered face.

Let Ψ_{prim} be the primitive elements of the fibered cone discussed in Section 3. Let

$$\begin{aligned} d_g &= \min\{\lambda(\psi) : \psi \in \Psi_{\mathrm{prim}}, \text{genus of } \psi \text{ is } g\}, \text{and} \\ d_g^+ &= \min\{\lambda(\psi) : \psi \in \Psi_{\mathrm{prim}}, \text{genus of } \psi \text{ is } g, \text{ the monodromy of } \psi \text{ is orientable}\}. \end{aligned}$$

In this section, we finish the proofs of Theorem 1.4 and Theorem 1.5 and their consequences by determining d_g and d_g^+.

Proposition 4.1. *Let $(a, b) \in \Psi_{prim}$. Then*

$$\lambda_{(a,b)} < \lambda_{(a',b')}$$

if either

(1) *$|a| < |a'|$ and $|b| = |b'|$; or*
(2) *$|a| = |a'|$ and $|b| > |b'|$.*

Proof. One compares the slopes of rays from the origin to (a, b) and (a', b'). The claim follows from Theorem 2.2. □

Proposition 4.2. *For $b \geq 3$, we have*

$$\lambda_{(1,b)} \geq \lambda_{(3,b+1)},$$

with equality when $b = 3$.

Proof. Let $\lambda = \lambda_{(3,b+1)}$. We will show that $LT_{(1,b)}(\lambda) < 0$. Multiplying by λ^2 and using the fact that $LT_{(3,b+1)}(\lambda) = 0$ gives

$$\begin{aligned} \lambda^2 LT_{(1,b)}(\lambda) &= \lambda^2 LT_{(1,b)}(\lambda) - LT_{(3,b+1)}(\lambda) \\ &= \lambda^{b+4} - \lambda^{b+3} - \lambda^{b+2} + \lambda^{b-2} + \lambda^2 - 1 \\ &= (\lambda - 1)(\lambda^{b+3} - \lambda^{b-2}(\lambda^3 + \lambda^2 + \lambda + 1) + \lambda + 1) \\ &= (\lambda - 1)\lambda^{b-2}[\lambda^5 - \lambda^3 - \lambda^2 - \lambda - 1 + \lambda^{2-b}(\lambda + 1)]. \end{aligned}$$

Thus, it is enough to show that for $\lambda > 1$ and $b > 3$

$$\lambda^5 - \lambda^3 - \lambda^2 - \lambda - 1 + \lambda^{2-b}(\lambda + 1) < 0.$$

Let C be the quantity on the left side of this inequality. Then

$$C < \lambda^5 - \lambda^3 - \lambda^2 = \lambda^2(\lambda^3 - \lambda - 1).$$

One can check that the right hand side is negative for

$$1 < \lambda < 1.3.$$

By Proposition 4.1, λ decreases as b increases. A check shows that

$$1 < \lambda_{(3,5)} < 1.3,$$

and hence $C < 0$ for $b \geq 4$. For $b = 3$, one checks directly that

$$\lambda_{(1,3)} = \lambda_{(3,4)}.$$

□

Remark. *The mapping class $\phi_{(1,3)}$ is defined on a genus 2 surface with four boundary components, with prong-type (3,1,1,1) and is not orientable. The mapping class $\phi_{(3,4)}$ is defined on a genus 3 surface with prong-type (4,4,4,4) and is orientable. By Proposition 4.2 these two examples have the same dilatation.*

Proposition 4.1 and Proposition 4.2 imply the following.

Proposition 4.3. *The sequences $\lambda_{(1,b)}$ and $\lambda_{(3,b)}$ satisfy:*

$$\lambda_{(1,b)} > \lambda_{(3,b+1)} > \lambda_{(1,b+1)}.$$

Table 3 describes the pairs $(a, b) \in \Psi_{\text{prim}}$ that give rise to the minima d_g and d_g^+ realized on M.

g mod 6	$\lambda(\phi_{(a,b)}) = d_g^+$, $\phi_{(a,b)}$ orientable	$\lambda(\phi_{(a,b)}) = d_g$
0	no example	$(3, g+1)$
1	$(3, g+1)$	$(3, g+1)$
2	$(1, g)$	$(1, g+1)$
3	$(3, g+1)$	$(3, g+1)$
4	$(1, g)$	$(3, g+1)$
5	$(1, g+1)$	$(1, g+1)$

TABLE 3. Pairs (a, b) giving smallest dilatations for $\phi \in \Phi(M_{\mathrm{sb}})$.

Proposition 4.4. *For $n \geq 2$,*

$$\lim_{n\to\infty} (\lambda_{(a,n)})^n = \frac{3+\sqrt{5}}{2},$$

for any fixed a.

Proof. The rays through the lattice points $(a, n) \in \Lambda_M$ on the fibered face of ψ converge to the ray through $(0, 1)$. □

Corollary 4.5. *For the minimal dilatations d_g and d_g^+ that are realized on M, we have*

$$\lim_{g\to\infty} (d_g)^g = \frac{3+\sqrt{5}}{2},$$

and

$$\lim_{\substack{g \to \infty \\ g \neq 0 (\mathrm{mod}\ 6)}} (d_g^+)^g = \frac{3+\sqrt{5}}{2}.$$

Table 3 and Corollary 3.9 complete the proofs of Theorem 1.4 and Theorem 1.5. A pictorial view of how the elements of Ψ giving the least dilatations for each genus up to 12 lie on a fibered cone of M is shown in Figure 5.

The results of this paper and those in [1, 14, 16] imply that for genus $g = 2, 3, 4, 5, 7,$ and 8,

$$\delta_g^+ = \lambda_{(a,b)}$$

where

$$(a,b) = \begin{cases} (1, g) & \text{if } g = 2, 3, 4, \text{ or } 8 \\ (1, g+1) & \text{if } g = 5 \\ (2, g+2) & \text{if } g = 7 \end{cases}$$

and

$$\delta_6^+ \geq \lambda_{(1,6)}.$$

These results suggest the following generalization to Question 1.3.

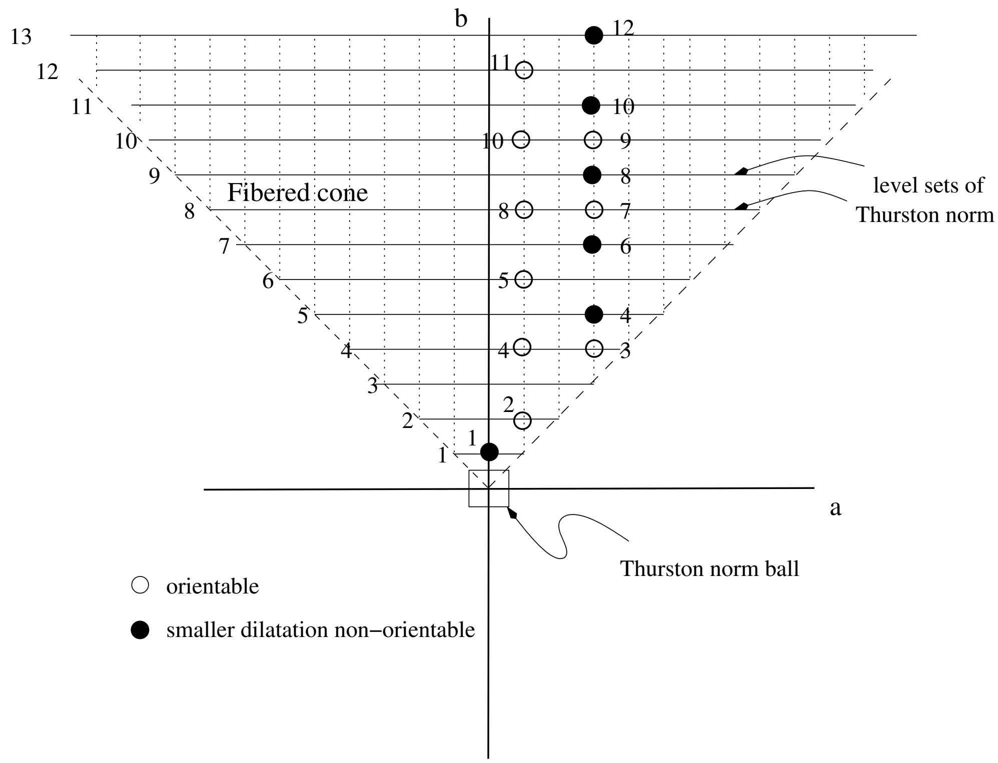

FIGURE 5. Minima for d_g and d_g^+ in genus $g = 1, \ldots, 12$.

Question 4.6. *For every $g \geq 2$, is it true that*

$$\delta_g^+ = \lambda_{(a,b)}$$

for some a, b with $b \geq g \geq a \geq 1$?

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