Minimum dilatation problem and quasi-periodicity conjecture

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This lecture is a survey of the minimum dilatation problem for pseudo-Anosov mapping classes. We will focus particularly on some outstanding questions and conjectures concerning the structure of mapping classes with asymptotically small dilatation using a sequence of mapping classes constructed by Penner as a motivating example.

We start with a brief summary of properties of pseudo-Anosov mapping classes (see also [Thu2], [FLP], [CB], [FM], [Far]).

Pseudo-Anosov mapping classes. Let $S = S_{g,n}$ be a compact oriented surface of genus g and n boundary components, where S has negative topological Euler characteristic. Consider the collection $\mathrm{MCG}(S)$ of isotopy classes of orientation preserving homeomorphisms $\phi : S \to S$ that fix the boundary of S pointwise. Such maps are called *mapping classes* and $\mathrm{MCG}(S)$ is the *mapping class group*. By a theorem of Nielsen and Thurston, for any $\phi \in \mathrm{MCG}(S)$, ϕ is periodic, reducible or pseudo-Anosov. These are defined as follows: a mapping class ϕ is *periodic* if it contains a representative whose kth power is the identity on S for some positive integer k; it is *reducible* if it preserves the isotopy class of a finite union of disjoint simple closed curves on S; it is *pseudo-Anosov* if there exist a pair of transverse measured singular foliations $(\mathcal{F}^{\pm}, \nu^{\pm})$ so that $\mathcal{F}^{\pm}$ are ϕ-invariant, and ϕ acts on the measures by $\phi_*(\nu^{\pm}) = \lambda^{\pm 1}\nu^{\pm}$. Here $\mathcal{F}^-$ is called the *stable foliation* and $\mathcal{F}^+$ is called the *unstable foliation* associated to ϕ. The stable and unstable foliations define a local Euclidean structure outside a finite set of *singularities* on S and the mapping class locally looks like a linear map:

$$(x, y) \mapsto (\lambda x, \frac{1}{\lambda} y)$$

outside the singular set. Equivalently (see [FM], Theorem 14.23), for any essential simple closed curve γ on S, and any fixed Riemannian metric defined on S, the lengths of iterations of ϕ on γ grows exponentially with growth rate λ independent from the choices of γ and the metric.

Properties of dilatations. The dilatations of pseudo-Anosov elements of $\mathrm{Mod}(S)$ are Perron algebraic units of degree bounded by $6g - 6 + 2n$ (see,[Pen], p. 447, and [AY]). That is, they are the largest real root λ of a reciprocal monic integer polynomial, so that all other roots have complex norm strictly smaller than λ. The roots of monic integer polynomials of bounded degree form a discrete set. Thus, in particular, the set of dilatations is a discrete set for fixed S. Define

$$\ell_{g,n} = \min\{\log(\lambda(\phi)) \mid \phi \in \mathrm{MCG}(S_{g,n})\}.$$

Problem 1 *Describe $\ell_{g,n}$ as a function of g and n.*

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So far, exact values for $\ell_{g,n}$ are known only for small g and n. For $(g,n) = (1,1)$ and $(0,4)$ the minimum can be found by analyzing hyperbolic elements of $\mathrm{SL}_2(\mathbb{Z})$, and we have

$$\ell_{1,1} = \ell_{0,4} = \log\left(\frac{3+\sqrt{5}}{2}\right).$$

For $(g,n) = (0,5), (0,6)$ and $(2,0)$, $\ell_{g,n}$ was found in [KLS], [HS], [CH], respectively, using train-track automata.. For $(g,n) = (0,5)$, ℓ_S is the logarithm of the largest root, or *house* of

$$x^4 - 2x^3 - 2x + 1$$

and for $(g,n) = (0,6)$ and $(2,0)$, ℓ_S is the logarithm of the house of

$$x^4 - x^3 - x^2 - x + 1.$$

The logarithm of the dilatation $\log(\lambda(\phi))$ can be interpreted as the minimal topological entropy of ϕ, and as the Teichmüller length of a corresponding geodesic in Moduli space. Penner studied the the asymptotic behavior of $\ell_{g,n}$, and proved the following.

Theorem 2 (Penner [Pen]) *The minimum dilatations for closed surfaces satisfy*

$$\ell_{g,0} \asymp \frac{1}{g}.$$

Penner proved this theorem by finding a universal lower bound for the house of Perron polynomials in terms of degree, giving

$$\ell_{g,n} \geq \frac{\log(2)}{12g - 12 + 4n}. \tag{1}$$

He found an upper bound for $\ell_{g,0}$ by constructing an explicit example, which we will now describe.

Penner's example. Consider the surface $S_{g,2}$ of genus g with $n = 2$ boundary components shown in Figure 1. The mapping class ϕ_g is the composition

$$\phi_g = r_g \delta_{c_g} \delta_{b_g}^{-1} \delta_{a_g}.$$

Where for any essential simple closed curve a, δ_a denotes the right Dehn twist centered at a.

Penner showed that

$$\lambda(\phi_g)^g \leq 11.$$

The two boundary components can be filled in to form singularities for the invariant stable and unstable foliations, defining a mapping class on a closed surface of genus g. Together with (1) this proves Theorem 2. Later in this article, we will show how to view Penner's example as a sequence on a fibered face, and find an explicit formula for the dilatations.

Pseudo-Anosov mapping classes with asymptotically small dilatations. For surfaces $S_{0,n}$, we have the following result.

Theorem 3 ([HK]) *The minimum dilatation for spherical pseudo-Anosov mapping classes satisfies*

$$\ell_{0,n} \asymp \frac{1}{n}.$$

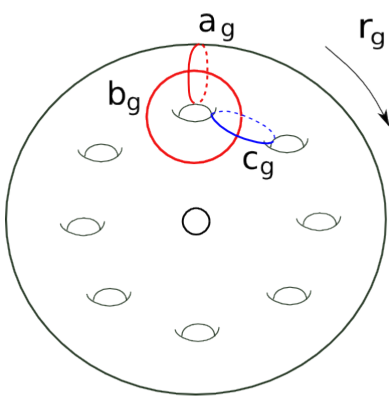

Figure 1: Penner's sequence of pseudo-Anosov maps with asymptotically small dilatation

Tsai asked the following.

Question 4 ([Tsa]) *What is the asymptotic behavior of $\ell_{g,n}$ where (g, n) varies along rays in the $(g, n)-$plane?*

For $g = 1$, Tsai showed that

$$\ell_{1,n} \asymp \frac{1}{n}$$

([Tsa], Appendix)

Valdivia answered Question 4 for rays emanating from the origin with positive rational slope.

Theorem 5 ([Val]) *Let $r > 0$ be a rational number. Then*

$$\ell_{g,n} \asymp \frac{1}{|\chi(S_{g,n})|},$$

where (g, n) varies along a ray of slope r through the origin, and $\chi(S_{g,n})$ is the topological Euler characteristic of $S_{g,n}$.

Tsai also showed that the behavior is different for surfaces with fixed genus $g \geq 2$.

Theorem 6 ([Tsa], Theorem 1.1) *For fixed $g \geq 2$,*

$$\ell_{g,n} \asymp \frac{\log(n)}{n}.$$

This leads to the question:

Question 7 *Does*

$$\overline{d}_r = \liminf |\chi(S_{g,n})|\ell_{g,n}$$

where (g, n) is contained in a ray of positive rational slope r through the origin, extend to a continuous function of r that goes to infinity as r approaches 0 from above?

Normalized dilatations. Considering the asymptotic results mentioned above, a more convenient invariant of a pseudo-Anosov mapping class $\phi : S \to S$ could be the *normalized dilatation*

$$\overline{\lambda}(\phi) = \lambda(\phi)^{|\chi(S)|}.$$

Problem 8 *For which naturally defined collections of mapping classes $\mathcal{F} \subset \cup_{g,n} MCG(S_{g,n})$, are there elements $\phi_k : S_k \to S_k \in \mathcal{F}$ such that $|\chi(S_k)| \to \infty$ and $\overline{\lambda}(\phi_k)$ is bounded?*

Remark. A weakness in the definition of normalized dilatation is that the dilatation of a pseudo-Anosov mapping class $\phi : S \to S$ only depends on the action of ϕ on the complement of the singular set of ϕ. Thus, it may be more natural to normalize dilatation using the topological Euler characteristic of the complement of the singular set on S, rather than on S itself, especially in light of Proposition 17 and Theorem 16. For the purpose of settling problems such as Problem 8, however, this nuance is not crucial, since we are only looking for an upper bound on $\overline{\lambda}$.

Minimization problem for special subsets of the mapping class group. Farb, Leininger, and Margalit show in [FLM1] that algebraic complexity can imply dynamical complexity. If $\mathcal{F}$ is the union of pseudo-Anosov elements in the Torelli subgroups of $\mathrm{MCG}(S)$, where S is a closed surface, then λ is bounded from below by a number strictly greater than 1 [FLM1]. More generally, let $\mathcal{N}_k$ be the kernel of the map

$$\mathrm{MCG}(S) \to \mathrm{Out}(\Gamma/\Gamma_k),$$

where $\Gamma = \pi_1(S)$ is the fundamental group and Γ_k is the kth term of the lower central series of Γ.

Theorem 9 (Farb-Leininger-Margalit [FLM1]) *There are constants $M_k > m_k > 1$ so that if $\phi \in \mathcal{N}_k$ is pseudo-Anosov, then*

$$m_k < \lambda(\phi) < M_k,$$

and $m_k \to \infty$ as $k \to \infty$.

The addition of geometric constraints can also constrain the range of possible dilatation (cf. [BL]).

There are, however, some positive answers to Problem 8 for other restrictive subsets of mapping classes. For example, for a closed surface S, let $\mathrm{hMCG}(S)$ the the collection of handlebody mapping classes, i.e., mapping classes $\phi : S \to S$ that extend to a handlebody whose boundary is S. Given a mapping class $\phi : S \to S$, there is an induced map $\phi_* : H_1(S;\mathbb{R}) \to H_1(S;\mathbb{R})$ on first homology. The spectral radius of ϕ_* is called the *homological dilatation* of ϕ.

Theorem 10 ([Hir3]) *There are mapping classes $\phi_g \in hMCG(S_{g,0})$ such that*

$$\overline{\lambda}(\phi_g) \leq C,$$

for some bound C that is independent of g. Furthermore, these examples can be taken so that their homological dilatations are trivial.

The second part of Theorem 10 contrasts with the result of Theorem 9.

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Fibered faces for hyperbolic 3-manifolds. The proof of Theorem 5 relies on a variation of Penner's sequence, while Theorem 6 is based on examples in [HK]. Both types of examples can be thought of as taking a single pseudo-Anosov mapping class, and "deforming it" to mapping classes on surfaces with large topological Euler characteristic (in absolute value). The idea of deformations of mapping classes can be made precise using the language of fibered faces

Let M be a hyperbolic 3-manifold, and let $\Psi(M)$ be the collection of fibrations $\psi : M \to S^1$. Then $\Psi(M)$ sits naturally inside $H^1(M;\mathbb{Z})$, which in turn forms a lattice inside $H^1(M;\mathbb{R})$. This is because fibrations determine homomorphisms from $\pi_1(M) \to \pi_1(S^1)$. Since $\pi_1(S^1)$ is the infinite cyclic group, and $H_1(M;\mathbb{Z})$ is the abelianization of $\pi_1(M)$, these homomorphisms factor through maps $H_1(M;\mathbb{Z}) \to \mathbb{Z}$. Making the identification

$$H^1(M;\mathbb{Z}) = \mathrm{Hom}(H_1(M;\mathbb{Z}),\mathbb{Z}),$$

we have an injective map $\Psi(M) \to H^1(M;\mathbb{Z})$.

In [Thu1], Thurston defines a norm $|| \ ||$ for $H^1(M;\mathbb{R})$ whose restriction to integral elements $\alpha \in H^1(M;\mathbb{Z})$ equals $|\chi(S)$, where S is a minimal surface in M representing α. When $\psi \in H^1(M:\mathbb{R})$ corresponds to a fibration, $||\psi|| = |\chi(S)|$, where S is the fiber of ψ. (For non-hypbolic 3-manifold M, Thurston makes a similar definition of a semi-norm on $H^1(M;|R)$.)

Theorem 11 (Thurston [Thu1]) *The unit ball with respect to the Thurston norm is a convex polyhedron. For every top dimensional face F, let $\Psi(M,F) = \Psi(M) \cap F \cdot \mathbb{R}^+$. Then $\Psi(M,F)$ s either empty or*

$$\Psi(M,F) = H^1(M;\mathbb{Z}) \cap F \cdot \mathbb{R}^+.$$

In the latter case, F is called a *fibered face*, and $F \cdot \mathbb{R}^+$ is the associated *fibered cone*. A *primitive element* of $\Psi(M,F)$ is a fibration with connected fibers. These correspond to integral points on rational rays through F with minimal Thurston norm. Thus, there are canonical one-to-one correspondences

$$\begin{aligned}\{\text{primitive elements of } \Psi(M,F)\} &\leftrightarrow \{\text{elements of } \Phi(M,F) \text{ with connected fibers}\} \\ &\leftrightarrow \{\text{rays through the origin passing through rational points on } F\}.\end{aligned}$$

For the rest of this paper, we will use the notation ψ to denote a fibration with connected fibers, $\overline{\psi} = \frac{\psi}{||\psi||}$ to denote the corresponding rational element on a fibered face, and ϕ to denote the monodromy of ψ.

We can think of $\Psi(M,F)$ and $\Phi(M,F)$ as deformation spaces of fibrations and mapping classes. The following is an outcome of the strong relation between the dynamics of elements of $\Phi(M,F)$.

Corollary 12 *Let b be the first Betti number of M. Then any fibration of M belongs to a $b-1$ dimensional deformation space. If $b \geq 2$, then the monodromy of any fibration of M has deformations defined on surfaces so that the absolute value of their topological Euler characteristic is arbitrarily large.*

Theorem 13 (Fried [Fri], McMullen [McM]) *Given a fibered face F, there is a convex function*

$$L : F \to \mathbb{R}$$

so that for all $\psi \in \Psi(M,F)$, with monodromy ϕ,

$$L(\overline{\psi}) = \lambda(\phi)^{||\psi||}.$$

Furthermore, this function goes to infinity near any point on the boundary of F.

Corollary 14 *For any sequence $\phi_n \in \Phi(M,F)$, and corresponding fibrations $\psi_n \in \Psi(M,F)$, if $\overline{\psi}_n$ is a convergent sequence on F, then the normalized dilatation $\overline{\lambda}(\phi_n)$ is bounded.*

Thus, as is noted in [McM], fibered faces are a rich source of examples of mapping classes with asymptotically small dilatations.

Quasi-periodicity conjecture. Motivated by Penner's original examples, Penner, and later Farb, Leininger and Margalit in joint work conjectured the following.

Conjecture 15 (Penner, Farb-Leininger-Margalit) *Let $\phi : S \to S$ be a mapping class with normalized dilatation satisfying*

$$\lambda(\phi)^{|\chi(S)|} < P. \tag{2}$$

Is there a bound K depending only on P, so that $\phi = r \circ f$, where $r : S \to S$ is periodic, and $f : S \to S$ is identity outside of some $Y \subset S$ such that $|\chi(Y)| < K$?

If Y is minimal, we call (Y, f) the *quasi-periodic support* of ϕ. Farb, Leininger and Margalit made fundamental progress toward characterizing asymptotically small dilatation mapping classes by showing the following.

Theorem 16 (Farb-Leininger-Margalit [FLM2]) *For each $P > 1$, there is a finite collection of hyperbolic 3-manifold Ω_P so that for any ϕ satisfying (2), the mapping torus of ϕ^0 is contained in Ω_P, where ϕ^0 is the restriction of ϕ to the complement of the singular locus of ϕ in S.*

The mapping class ϕ is related to its restriction ϕ^0 in the following way.

Proposition 17 (e.g., [HK], Lemma 2.6) *Let p be a singularity of ϕ and let $\mathcal{O}_p$ be the orbit of p under the map ϕ. Then the restriction ϕ^0 of ϕ to $S \setminus \mathcal{O}_p$ is also pseudo-Anosov and $\lambda(\phi^0) = \lambda(\phi)$.*

A mapping class pair (Y, f) is a mapping class $f : Y \to Y$. A *covering* of mapping classes

$$(Y_1, f_1) \to (Y_0, f_0)$$

is a covering $Y_1 \to Y_0$ so that the following diagram commutes.

$$\begin{tikzcd} Y_1 \arrow[r, "f_1"] \arrow[d] & Y_1 \arrow[d] \\ Y_0 \arrow[r, "f_0"] & Y_0. \end{tikzcd}$$

Let $\mathcal{F}$ be a family of pseudo-Anosov mapping classes (S, ϕ) where the surfaces appearing in $\mathcal{F}$ realize arbitrarily large topological Euler characteristics. We say $\mathcal{F}$ is *quasi-periodic* if for any mapping class $\phi \in \mathcal{F}$, $\phi = r \circ \phi$, where the quasi-periodic support (Y, f) of ϕ is finite covered by $(Z, \widetilde{f})$. A family is *strongly quasi-periodic* if if the quasi-periodic support of $\phi \in \mathcal{F}$ is fixed, and there is no need to pass to a finite cover.

Question 18 *Is it true that for any fibered face F, and any compact subset $K \subset F$, the mapping classes associated to rational points on F form quasi-periodic families?*

By Theorem 16 an affirmative answer to Question 18 implies Conjecture 15.

Constructions of quasi-periodic families.

Start with a pseudo-Anosov mapping class $\phi : S \to S$, and a ϕ-invariant class $\tau \subset H_1(S, \partial S; \mathbb{Z})$. The triple (S, ϕ, τ) determines a linear section of the fibered face F of the mapping torus M of ϕ such that $\phi \in \Phi(M, F)$.

The relative homology class τ determines a cyclic covering

$$\rho : \widetilde{S} \to S,$$

defined by the epimorphism $H_1(S;\mathbb{Z}) \to \mathbb{Z}$, sending a closed loop γ to its algebraic intersection with τ. Let M be a hyperbolic 3-manifold and F a fibered face. For any subset $A \subset F$, let $\Phi(M, A)$ be the collection of monodromies corresponding to rational points in A.

Theorem 19 ([Hir2]) *Let $\zeta : \widetilde{S} \to \widetilde{S}$ be a generator for the (cyclic) group of covering automorphisms of $\widetilde{S}$ over S. Suppose there are mapping classes $\widehat{\phi}_i : \widetilde{S} \to \widetilde{S}$ with compact support so that*

$$\zeta \circ \widehat{\phi}_i = \widehat{\phi}_{i+1} \circ \zeta.$$

for all $i \in \mathbb{Z}$. Then the linear section L of the fibered face F defined by (S, ϕ, τ) has the property that $\Phi(M, L)$ is quasi-periodic, and for some open neighborhood U of $\overline{\psi}$ on F, the collection $\Phi(M, L \cap U)$ is a strongly quasi-periodic family.

Wedge triples. A triple (S, ϕ, τ) that satisfies the hypotheses of Theorem 19 is said to have *compact lifting*. It is called a *wedge-triple* if $\phi = \delta \circ \eta$, where δ is a Dehn twist along a curve γ, where the algebraic intersection of γ with τ is trivial, and η is a reducible mapping class on S that fixed τ. The reason for the definition is because η lifts to a fundamental domain for the covering automorphisms of ρ, which can be thought of as a wedge, and δ is a Dehn twist on a "connecting curve" that crosses over the fundamental domains. Penner's original example comes from the wedge-triple shown in Figure 2 (compare Figure 1). Here, the mapping class ϕ is the composition of Dehn twists

$$\phi = d_c d_b^{-1} d_a.$$

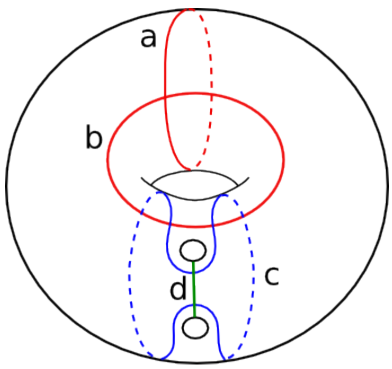

Figure 2: Triple defining Penner's original example

Theorem 20 ([Hir2]) *If (S, ϕ, τ) is a wedge triple, then it has a compact lifting.*

Corollary 21 *Penner's examples are the monodromies of fibrations corresponding to points on a linear section of a fibered face.*

Teichmüller polynomial. One advantage of realizing mapping classes as monodromies of fibrations on a fibered face is that this provides a way to compute the dilatations explicitly. Given a polynomial $p(x)$, let $|p|$ be the maximum complex norm of roots of p. If G is a group,

$$f = \sum_{g \in G} a_g g \in \mathbb{Z}G$$

an element of the group ring, and $\psi : G \to \mathbb{Z}$ is a homomorphism, then the specialization of f at ψ is defined by

$$f^\psi = \sum_{g \in G} a_g x^{\psi(g)}.$$

Theorem 22 (McMullen [McM]) *Given a hyperbolic 3-manifold M and a fibered face F, there is an element $\Theta \in \mathbb{Z}G$, where $G = H_1(M;\mathbb{Z})$, so that for all $\psi in \Psi(M, F)$,*

$$\lambda(\phi) = |\Theta^\psi|$$

where ϕ is the monodromy of ψ, and $|\Theta^\psi|$ is the house of the specialization.

Let (S, ϕ, τ) be the wedge-triple for Penner's example. One can find coordinates ψ, ξ for $H^1(M;\mathbb{Z})$, where $M = M_\phi$ is the mapping torus for ϕ, so that the Teichmüller polynomial is given by

$$\Theta = u^2 - u(5 + t + t^{-1}) + 1.$$

It follows that if ϕ_g is the element of the Penner sequence with genus g shown in Figure 1, then

$$\lambda(\phi_g) = |x^{2g} - x^{g+1} - 5x^g - x^{g-1} + 1|.$$

One can veriify that

$$\overline{\lambda}(\phi_g) \to \left(\frac{7 + 3\sqrt{5}}{2}\right)^2 \approx 46.9787.$$

Remark: The smallest known accumulation point for normalized dilatations of pseudo-Anosov mapping classes is

$$\left(\frac{3 + \sqrt{5}}{2}\right)^2 \approx 6.8541$$

see [Hir1], [AD] [KT] [KKT].

We end with a question:

Question 23 *Are there mapping classes with bounded normalized dilatations that are not quasi-periodic? If so what expanded definition of quasi-periodicity is needed to include such examples?*

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